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A

NEW GENERAL THEORY

OF

THE TEETH OF WHEELS.

BY

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TO THE

PRESIDENT, COUNCIL, AND MEMBERS

OF THE

ROYAL SCOTTISH SOCIETY OF ARTS,

AS A TOKEN OF CHERISHED REMEMBRANCE,

THESE PAGES ARE INSCRIBED BY

THEIR OBEDIENT SERVANT,

THE AUTHOR.

CONSTANTINOPLE,
December 1860.

INTRODUCTION.

In the year 1837, a short paper, containing the general principle of the method of investigation followed in the present Work, was read before the Royal Scottish Society of Arts. Being intended for publication in the Society's Transactions, it was withdrawn for the purpose of having incorporated with it some new subjects which had subsequently presented themselves; but, on account of the more pressing avocations of the writer, its reproduction was delayed from time to time. Once and again, however, the investigation was resumed, and at each revision the power and the fertility of the method were exemplified by the opening up of new lines of inquiry, until, having lost almost all trace of its original form, the paper became far too extensive for its first destination.

The method of research therein explained, collecting under one point of view all that was previously known of the theory of toothed wheels, has brought to light important general principles, and has guided the application of these to the solution of problems which, before, could hardly have been entered upon. Under these circumstances the author takes some blame to himself for having so long postponed the publication of a theory which, in other hands, might, by this time, have received a much greater development. He has endeavoured to atone for the unwilling delay by increased care in the elucidation of the subject; and his regret has been somewhat lessened by the fact that, having frequently recast the essay for the last time, he has been obliged, by the accession of new ideas, to recast it once more.

The theory of toothed wheels, involving, from its very nature, the doctrines of contact and of variations, necessarily belongs to the higher department of the mathematics, and requires the assistance of the fluxional calculus. On this account, it is impossible to render all our investigations intelligible to those who have proceeded no farther in their studies than to elementary algebra and geometry: yet, wherever it was found possible, the aid of the higher calculus has been dispensed with, and the results have been enunciated in such a way as to be intelligible to the great bulk of students of engineering. May the perusal incite to renewed exertion those who still hesitate at the threshold of the differential and integral calculus!

The study of the proper form for the teeth of wheels is forced upon us by the frequent occurrence of wheel-work in machinery. In the earlier machines it was considered enough that one tooth had not ceased its action until another tooth had become engaged; but by degrees, as wheel-work came to be applied to more and more delicate purposes, a relative uniform motion of the two axes came to be desirable. Römer, the celebrated Danish astronomer, proposed, in the year 1674, the *epicycloid* as the proper form for the tooth. Since that time the epicycloid and its obvious extension, the involute of the circle, have been regarded as the true forms for the teeth of wheels; yet, although these curves satisfy, geometrically, the condition of contact, they are very far indeed from satisfying the wants of the machine maker. In each of them we find points of reflexure which separate the curve into branches, whose convex and concave sides are alternately turned to the action, so that the tooth which has glided along the convex branch effaces, by its subsequent motion, all traces of the concave one. Hence arises the necessity for compounding the outline of the wheel from parts of heterogeneous curves often presenting awkward junctions, and at all times impeding the application of general formulæ. However carefully the parts of two curves may be run into each other at their point of junction, there must always be some abruptness in the transition from the one to the other. Though their extremities agree in direction so as to have a common tangent, their curvatures may be different; and even although their osculating radii be alike, the laws according to which the curvature changes cannot be so; so that the only way to obtain the character of smooth transition in every possible respect, is to have one analytic curve along the whole outline, and even from among these we must exclude all curves whose osculating radius at any time becomes zero.

The whole subject may be condensed into one proposition in this way. Having arbitrarily prescribed the form of one toothed wheel A, and communicated to it, by any imaginable contrivance, the requisite angular velocity; let a blank disc B turn alongside of A also with its appropriate motion. Trace now upon the blank surface the successive instantaneous outlines of the wheel A, and draw a curve enveloping all these traces; then this enveloping curve is the form for the wheel B. In this way, whatever may have been assumed for the form of the wheel A, the delineation of the second wheel B is always *geometrically* possible. Having now obtained the form of B, let the wheel A be removed, and its place supplied by a blank disc: and, treating B as the primary wheel, draw its traces upon the new blank; the curve enveloping all these traces must clearly be a copy of the original form A. Hence the two forms A and B are *conjugate*, neither of them having, as it were, any claim to be regarded as the original; and thus we are led to consider A and B not as derived the one from the other, but as both deduced from some *datum* still more elementary in its nature.

While the two forms A and B continue to revolve with their appropriate velocities, their contours remain in contact, but the position of the point of contact shifts with the motion. If, then, the path of this point of contact could have been prescribed beforehand, and the law of its motion therein discovered, that motion, combined with the angular velocities of the two discs, would have traced out the shapes of the two wheels A and B, which would then have been homologously derived from one primary.

Suppose now a tracing point P to describe, according to some yet unknown law, an orbit in the vicinity of two revolving discs A and B, it would trace upon each of those discs a curved line: these curves would always meet at P, since that point is common to them both, and meeting, they might intersect or touch each other according to the arrangement of the motions. For the purposes of engrainage, it is requisite that the two traces be continually tangent to each other, and it becomes necessary to subject the motion of the point P to some condition which may ensure the contact of the two traces. The discovery of this essential condition is our first business.

On investigating the law to which the motion of the point P must be subjected, in order that its traces upon two uniformly revolving discs A and B may be tangents to each other, we arrive at this important, and, so far as I know, hitherto unnoticed porism, *that the motion of the tracing point is independent of the positions of the centres, being dependent only on its position in respect of the pitch point and on the common velocity of the pitch lines.* In other words, if we have a multitude of wheels whose centres are in one straight line, and which have a common pitch point, that motion of P which suits any one pair of these wheels will suit every other pair of them.

On still farther examining the condition of contact, we find that it merely establishes a law between the form of the path of the tracing point and the velocity of the motion along it, leaving the figure and dimensions of the orbit to be determined by other considerations. For each species of orbit which we may assume we have thus a distinct system of wheel-work, and we come to regard wheels not as adapted to each other in pairs, but as grouped together in systems.

When we combine the essential condition of contact with these other two requisites, *finitude in the extent of the orbit and complete interchangeability among the wheels of one system,* we obtain in general, for the path of the tracing point, orbits whose type is the *hour-glass curve,* a curve which is familiar to engineers as that described by the attachment of the piston-rod in Watt's parallel motion, and which is also exhibited by the vibration of a straight wire whose breadth is double of its thickness.

On assuming this curve, the simplest of its class, for the orbit of the tracing point, we obtain a series of beautiful results. Thus, taking the limiting angle of the curve at 45° or less, we have the orbit of one solitary point of contact: but, on augmenting this angle, other contacts make their appearance, always, however, in pairs; and thus we are enabled to determine what must be the ratio between the length and the breadth of the orbit, in order that there may be always one, three, five, or any other odd number of points in contact at once. Of these contacts some, near the points of the teeth, are comparatively valueless ; others near the shoulders are properly the working ones, and these are always one more in number than the former. If, then, we propose to construct a system of wheels which shall always have two teeth fully engaged, we must have two useful contacts forwards, and, to match them, two backwards, in all four contacts on the sides, and these, along with the three complementary ones, make up the total number of seven contacts : therefore we must select that form of the orbit which gives continually seven points in contact at a time.

Having made our selection and begun to delineate the wheels, we find that the outlines for low numbers are much convoluted, so that the contacts, though geometrically existing, are mechanically deficient. Thus with seven contacts, all wheels above 33 teeth have their contacts complete ; but on descending to 32 teeth, convolutions appear at the points, and the action there ceases to be mechanically complete ; the deficient contacts, however, are complementary ones, and the want of them is not felt, only the teeth, instead of being rounded, have become pointed. It is not until we come so low as 18 leaves that any of the working contacts disappear ; even below that number the wheels will work, although not well ; but, when we descend to 11 leaves, convolutions make their appearance at the shoulders also, and the action ceases to be mechanically possible. Thus for each system of wheel-work we have three distinct metacentric distances, by which the smallest number of teeth which it is possible or advisable to use is determined.

The shapes obtained by help of the hour-glass curve are demonstrably the simplest analytic forms which can satisfy the conditions of contact and of permutability, and therefore their theory has been examined at great length. But mere simplicity in the analytic definition of the curve, although it have its value, is not of paramount importance to the machine-maker ; and hence we eagerly followed up some views of ulterior advantages possibly possessed by other systems, which this examination laid open to us.

On applying the same methods to the involute and epicycloidal teeth, the principles which limit the number of their contacts were at once unveiled. Thus with involute teeth touching at seven places, the wheel of 17 has its contacts perfect, while with cycloidal teeth we may go as low as 11. These numbers, contrasted with the 19 of the hour-glass system, shew a considerable advantage, particularly in favour of the epicycloid. Yet there are two drawbacks ; one that the teeth of the small wheels are undercut, the other, that the radius of curvature changes from + 0 to — 0 at the junction of the two epicycloids, and that therefore the practical shaping of the teeth there is impossible ; and, to make matters worse, this imperfection occurs just where the action otherwise is at its best.

Thus arose the problem, *To discover that system of engrainage which, with a given number of contacts, admits of the smallest pinion.* This question being too formidable to be attacked in all its generality, we were content to seek for some modification of the hour-glass curve which may permit of lower-numbered pinions. On adding a new term to the equation of that curve, it was found that the least of the three metacentric distances increased while the others decreased, and the value of this new term was then determined, so as to make two of the metacentres coalesce. In this way two distinct modifications were obtained : in the first the convolutions appear simultaneously at the points and at the shoulders of the teeth, the smallest pinion being numbered 17 ; in the second, at the appearance of convolutions on the shoulders, the useful contacts at the points just begin to be trenched upon, the smallest possible pinion having 14 leaves. This latter modification gives us low-numbered wheels, presents one continuous analytic curve for the outline of the wheel, thereby avoiding all sudden transitions, and permits all wheels of the same pitch to be interchanged in any way, therefore it may fairly be regarded as in every respect the most eligible.

The dimensions of the various orbits and the co-ordinates of the tracing points have been computed with the most scrupulous care, and with all the precision which seven-place logarithmic tables could give. By help of these any one may project for himself the form of a wheel according to whatever system he may select. In the course of this work it has been necessary to resolve many equations of the form $x + m . \sin x + n . \sin 3x = \sigma$, and to obtain not only one, but all the possible roots of each equation; while the discovery of the dimensions of the modified hour-glass curves has required an amount of labour which nothing but the hope of obtaining a valuable result could have induced one to undergo. Yet, notwithstanding the enormous toil, I am confident that no material error will be found among the tabulated results, and that these may be used with every reliance on their accuracy. While carrying on the computations, I have, a thousand times, regretted the present inconvenient subdivision of the circle: with tables constructed according to the decimal subdivision the labour would have been much reduced.

Besides the form of the tooth, other topics have been discussed. The amount of friction has, perhaps for the first time, been rigorously examined, and the statical and phoronomic effects of malformation have been pointed out. The concluding article on the computation of trains of wheels will prove, it is hoped, a valuable aid to those engaged in the laborious process of constructing orreries: for the greater part of it the author believes that he can claim the merit of originality.

CONSTANTINOPLE,
17*th December* 1850.

THEORY
OF THE
TEETH OF WHEELS.

IN its most unrestricted acceptation, the theory of the teeth of wheels ought to contain the solution of the following problem:—

Two nuclei move in space according to prescribed laws; it is required to surround these nuclei by solids whose surfaces may continually touch each other.

And, if it were our object to attain to the solution of all questions of engrainage with the least possible amount of labour, we should at once proceed to the solution of the general problem, and thence descend to the particular cases by changing the values of the arbitraries which had entered into the formulæ. But, by such a proceeding, we should drag those whose object is to study the simpler and more practical cases, through all the intricacies of the most complex one; nor would we obtain, in return for this inconvenience, any increased facility in the elucidation of the subject: besides, the very simplest of all the cases is by far the most important. On this account we prefer to begin with the theory of plain gear, so as to allow the ideas gradually to develope themselves in a natural manner.

The motion of a solid body may always be decomposed into two, one of translation, the other of rotation round an instantaneous axis. By supposing the motion of translation to be zero, and the instantaneous axis to be permanent, we obtain the first restriction of the unlimited problem; thus

Two axes, situate any how in space, rotate according to prescribed laws; it is required to surround these by solids which may continually touch each other.

In the ordinary cases of wheel-work the angular velocities are proportional, so that, if one axis be supposed to turn uniformly, the other axis ought also to turn with a uniform velocity; thus we have the problem again restricted to this—

Two axes, situate any how in space, turn with given velocities; it is required to surround these axes with solids which may continually touch each other.

If the two axes neither meet nor be parallel to each other, that is, if they be not both in one plane, we have the case of what is called *skewed gear.* When the axes meet, we have the case of *bevelled gear;* and, when the axes are parallel to each other, that of *plain gear.*

According to our plan, the last case, being the most simple, falls to be considered first; yet, in order to preserve uniformity in our notation, we shall glance at the case of skewed gear, so as to prepare for our future generalizations.

A *a*, B *b*, being the two axes of rotation, situate any how in space, their positions in reference to a system of rectangular co-ordinates may be indicated by the equations

$$\frac{x_a - x_A}{\cos b \,.\, \sin a} = \frac{y_a - y_A}{\sin b \,.\, \sin a} = \frac{z_a - z_A}{\cos a};$$

$$\frac{x_b - x_B}{\cos \beta \,.\, \sin \alpha} = \frac{y_b - y_B}{\sin \beta \,.\, \sin \alpha} = \frac{z_b - z_B}{\cos \alpha};$$

in which a and α are the inclinations of the axes to the direction Z, b and β their azimuths, counted from the direction X towards the direction Y.

Now, whatever may be the relative positions of two lines, it is always possible to find a plane parallel to them both, and a line, normal to that plane, cutting them both at right angles. Being at liberty to choose the most convenient position for our system of co-ordinates, we may place the plane X Z parallel to the two axes, and make the line Y coincide with the line which meets them both at right angles, without thereby impairing in the least degree the absolute generality of the data; that is to say, we may suppose each of the quantities b, X_A, Z_A; β, X_B, Y_B to be zero, when the above equations become

$$\left.\begin{array}{l} z_a \tan a = x_a;\ y_a = y_A; \\ z_b \tan \alpha = x_b;\ y_b = y_B; \end{array}\right\} \quad (3)$$

which we shall afterwards use when treating of skewed gear.

To make the axes intersect each other, we have only to put $y_A = y_B$, and then we are at liberty to place the origin at A, which gives $y_A = y_B = 0$; hence we obtain the equations,

$$\left.\begin{array}{l} z_a \tan\ a = x_a;\ y_a = 0; \\ z_b \tan\ \alpha = x_b;\ y_b = 0; \end{array}\right\} (2)$$

which will afterwards serve for bevelled gear.

Lastly, in order to render the axes parallel, we make $a = \alpha$, and, for convenience, each of them zero; that is, we bring the z's to be parallel with the axes, then

$$\left.\begin{array}{l} z_a = \text{indeterminate};\ x_a = 0;\ y_a = y_A \\ z_b = \text{indeterminate};\ x_b = 0;\ y_b = y_B \end{array}\right\} (1)$$

which are the equations of the axes A *a*, B *b* of two common wheels.

IN the case of parallel axes the ordinates z are indeterminate, and we may restrict our attention to the section made by any plane parallel to X Y, or to the section by X Y itself; that is, we may suppose z to be zero throughout. In this way we come to regard the wheels as mere discs, and the axes as points, by which means the language and the notation are greatly simplified, the enquiry becoming one in plane geometry. This restriction cannot affect the configuration of the section of the system of wheels by the plane $z = 0$; but it will afterwards be necessary to examine how two contiguous sections may be related to each other, and what may be the character of the surface which connects them.

Suppose two discs, properly toothed, to turn upon their centres A and B; their contours ought to be continually in contact, and the point or points of contact must pass along all those parts of the outline which are essential to the action. If then the motion of the points of contact be ascertained, and if that motion be afterwards combined with the proper angular motions of two blank discs which have been substituted for A and B, the true contours of the two wheels will be reproduced. Were we, however, to combine an arbitrarily assumed motion of a point P with the angular motions of the discs A and B, the curves thereby simultaneously traced out might intersect each other, instead of mutually touching; and hence our first business is

To discover according to what law the point P must move in order that its traces upon the two revolving discs A and B may touch each other.

The solution of this problem contains the germ of the whole theory of toothed wheels, and merits, therefore, the most attentive consideration. We shall present it first in the symbolical and afterwards in the geometrical form.

At the instant of time t, let the position of the tracing point P be denoted by the co-ordinates x_P and y_P, these being, of course, functions of the time t; and let the angular velocities of the two discs be v and φ, the motions being from the direction X into the direction Y. Then at the approximate instant $t + \partial t$ the co-ordinates will have become $x_P + \partial x_P$; $y_P + \partial y_P$, while the discs will have turned upon their centres A and B, through the small angles $v \partial t$ and $\varphi \partial t$.

The tracing point P having now assumed the approximate position P′ must correspond to two points which we shall call p' and π' upon the moveable blank discs; and we have to enquire what were the positions occupied by these points p' and π' at the previous instant t. For this purpose we cause the two discs to turn backwards through the minute angles $v \partial t$ and $\varphi \partial t$. Observing that these angles are infinitesimally small, and that, therefore, $\cos v \partial t = 1$, $\sin v \partial t = v \partial t$, and performing the requisite operation, we find, putting p and π for the desired positions,

$$\left.\begin{aligned} x_p - x_P &= \partial x_P + v \partial t \, (y_P - y_A) \\ y_p - y_P &= \partial y_P - v \partial t \, (x_P \qquad\;\;) \end{aligned}\right\} (4),$$

$$\left.\begin{aligned} x_\pi - x_P &= \partial x_P + \varphi \partial t \, (y_P - y_B) \\ y_\pi - y_P &= \partial y_P - \varphi \partial t \, (x_P \qquad\;\;) \end{aligned}\right\} (5).$$

Now Pp and Pπ are the elements of the two contours in the positions which they occupied at the instant of time t; and therefore, if the contours touch each other, the inclinations of these two elements must be identic, or the tangents of the angles which they make with the direction Y must be equal to each other; whence

$$\frac{x_p - x_P}{y_p - y_P} = \frac{x_\pi - x_P}{y_\pi - y_P}, \quad (6)$$

or, by substitution,

$$(v - \varphi)\,(y_P \partial y_P + x_P \partial x_P) + (\varphi y_B - v y_A)\, \partial y_P = (y_B - y_A)\, x_P \, v \varphi \partial t, \quad (7),$$

to which law the motion of the tracing point must be subjected, in order that its traces on the two discs may continually touch each other.

This equation may be greatly simplified in appearance by a proper transposition of the origin of the co-ordinates, which may be placed any where in the line A B: but the character of this transposition will be best seen from what follows.

Since equation (7) establishes only one relation between ∂x_P, ∂y_P, and ∂t, it leaves us at liberty to suppose any second condition not incompatible with the first, and thus the general problem admits of an infinity of solutions.

Let us for a moment suppose that the tracing point is confined to move in the line A B, and, for distinctness of language, let us change its name to Q; then we have $x_Q = 0$, $\partial x_Q = 0$, which, substituted in equation (7), give

$$\{ v (y_Q - y_A) - \varphi (y_Q - y_B) \} \partial y_Q = 0,$$

an equation which can be satisfied in two ways; either by supposing ∂y_Q itself to be zero, or by equating its coefficient to zero.

The first supposition, $\partial y_Q = 0$, places Q immoveable any where in the line AB, and brings us to the well-known geometrical theorem, that *if two circles meet in the straight line which joins their centres, they touch each other.*

The second supposition gives

$$\left.\begin{aligned} v (y_Q - y_A) &= \varphi (y_Q - y_B) \text{ or} \\ y_Q &= \frac{v y_A - \varphi y_B}{v - \varphi} \end{aligned}\right\} (8).$$

In the ordinary case of wheel-work v and φ are constant, therefore this equation also fixes Q immoveable in the line A B, and the wheels are again two circles touching each other. But, since the radii of these circles are $y_Q - y_A$ and $y_Q - y_B$, while v and φ are their angular velocities, the expressions $v (y_Q - y_A)$ and $\varphi (y_Q - y_B)$ give the linear velocities of the two circumferences. These linear velocities are, therefore, equal to each other, and in the same direction, so that the wheels roll upon each other.

Thus the point Q determined by equation (8) is what is technically called the *pitch-point*, and the circumferences traced by it are the *pitch-lines* which move with a common velocity. If we put q for the length of the pitch-lines which has passed during the time t, ∂q will represent that element of them which passes during the time ∂t, wherefore

$$\partial q = v (y_Q - y_A) \partial t = \varphi (y_Q - y_B) \partial t.$$

Now, if in equation (7) we first put $(v - \varphi) y_Q$ for $v y_A - \varphi y_B$, and then eliminate ∂t by help of the above expressions, we obtain

$$(y_P - y_Q) \partial y_P + x_P \partial x_P - x_P \partial q = 0, \ (9).$$

And lastly, if, *as is allowable in the case of uniform motion, but not otherwise*, we transfer the origin of the co-ordinates to Q, we have simply

$$y_P \partial y_P + x_P \partial x = x_P \partial q; \ (9 . a)$$

doubling and observing that $y_P^2 + x_P^2 = \text{Q P}^2$,

$$\partial . \text{Q P}^2 = 2 x_P \partial q; \ (9 . b)$$

putting ω for the angle which the line Q P makes with the direction X, and observing that $x_{\mathrm{P}} = \mathrm{Q\,P} \,.\, \cos\,\omega$, we have

$$\partial \,.\, \mathrm{Q\,P} = \partial q \,.\, \cos\,\omega\,; \quad (9\,.\,c)$$

hence this very simple enunciation ; *the tracing point must retire from the pitch point with a velocity equal to the product of the velocity of the pitch lines by the cosine of the angle* X Q P *which measures the obliquity of the action.*

The most important feature of equation (9) is, that the positions of the centres of the discs have entirely disappeared from it; the position of P in reference to the pitch point Q, and the velocity of the pitch line only remain. Hence we have this highly interesting and hitherto unnoticed porism, that *that motion of the tracing point which gives a true form to any pair of wheels, will give a true form to any other wheel having the same pitch point and the same velocity for its pitch line.* On the truth of this porism rests the possibility of constructing a system of wheels which may all work into each other.

Combining equation (9 . *a*) with (6) we find, for the value of the tangent of the angle which the element of the contour makes with the line Y, the expression

$$\frac{x_{\mathrm{p}} - x_{\mathrm{P}}}{y_{\mathrm{p}} - y_{\mathrm{P}}} = -\frac{y_{\mathrm{P}}}{x_{\mathrm{P}}}\,;$$

that is to say, the line which joins the pitch point with the point of contact is always normal to the contours of the teeth, and thus, putting friction out of view, the pressure upon the wheel is always directed from or towards the pitch point.

The truth of these theorems may be exhibited geometrically thus : Let, in figure 1, Plate I., A and B be the centres of two wheels which are geared into each other, and let Q be the position of the pitch point, so that two circles described with the radii A Q and B Q would roll upon each other with angular velocities inversely as these distances. Let also P be one position of the tracing point, and suppose that, while the pitch lines have moved through some extremely minute distance Qq, the tracing point has moved from P to P′. The question before us is to discover according to what law the motion P P′ must be regulated in order that the traces upon the two discs may touch each other. For this purpose we must ascertain where the two points on the discs with which P′ now coincides were at the beginning of the motion P P′ ; for this we cause the disc A to turn backwards through the angle q A Q and the disc B through the angle q B Q. Draw P′p perpendicular to A P′, and make A Q : A P′ : : Q q : P′p, then P′p, supposed to be infinitesimally small, represents the arc described by the point p during the motion Q q. Again, make B Q : B P′ : : Qq : P′π, and lay P′π

at right angles to BP', then p and π are the two points which, during the motion of the pitch lines over the distance Qq, have simultaneously come up to P'; therefore Pp must be an element of the contour of the wheel A and $P\pi$ the corresponding element of the outline of the wheel B. The nature of the question requires that these elements coincide in direction, or that P, p and π be in one straight line.

Having drawn $P'R$ perpendicular to AB, cut off $P'r$ equal to qQ; then we have by construction $AQ : AP' :: P'r : Pp$, while the angle $pP'r$ is evidently equal to $P'AQ$, and therefore the trigon $pP'r$ must be similar to $P'AQ$, consequently the angle $P'rp$ is equal to $P'QA$. In the same way the trigon $\pi P'r$ may be shewn to be similar to $P'BQ$ and the angle $P'r\pi$ to be equal $P'QB$; hence, since A, Q, B are in one straight line, p, r, π must also be in one straight line, which likewise must pass through P. Again, the sides $P'p$, $P'r$ of the trigon $pP'r$ being perpendicular respectively to AP' and AQ of the similar trigon PAQ, the third side pr must also be perpendicular to QP'; that is, the line QP' is normal to each of the curves.

Since the figure $Pp\pi rP'$ must be regarded as occupying a mere point in comparison with the figure ABP, the law of the motion of the tracing point P may be stated as follows:—

Having drawn through P a line perpendicular to QP, this line must be tangent to each of the contours: in it assume any proximate point r, and draw rP' parallel and equal to Qq, then P' may be taken as the proximate position of the tracing point. When the direction of the motion PP' is given, as when the tracing point is restricted to move in a given orbit, the intersection of that direction with a line drawn parallel to Pr at the distance Qq above it, will give the new position of P. Since the trigons $sP'r$ and $RP'Q$ are right angled and have a common angle, they are similar; wherefore $P'Q : P'R :: P'r : P's$, but $P's$ is clearly the increment of the distance QP, wherefore, in the language of the differential calculus,

$$QP \,.\, \partial QP = PR \,.\, \partial q \quad \text{or} \quad \partial QP = \cos QPR \,.\, \partial q,$$

which agrees exactly with equation 9.

Conversely, it can easily be proved that if two discs touch each other at the point P, and if PQ be drawn normal to the tangent surfaces, and be continued to meet the line of centres in Q, Q is the pitch point; that is to say, the angular velocities of the wheels at that instant are inversely proportional to the distances of their centres from Q.

When the tracing point P moves into a new position P', the only condition to which its motion must be subjected, is that the increase of its distance from Q, which increase is represented by sP', must be determined by the proportion $PQ : PR :: Qq : sP'$; the direction in which the motion takes place is a matter of indifference, and hence, excluding a straight line at right angles to QP, any path may be assigned for the tracing point, provided its velocity therein be properly regulated. Further, the law of the motion of P is entirely independent of the positions of the centres, and therefore the same motion which generates tangent traces upon the discs A and B will generate tangent traces upon any other discs whose centres are in the line A B, and which revolve with velocities inversely as the distances of their centres from the pitch point Q.

Hence, if from any wheel A we derive a series of conjugate wheels B, B′, B″, etc., and if, from the wheel B, we derive another series A′, A″, etc., each wheel of the series A will work truly with any one of the series B; nay more, any two of the series A will work together, the one being internal to the other, and the motion of the point of contact will be the same in every combination.

Although the equation (9) contain that condition which is essential to the true action of wheel-work, it does not determine completely the motion of the tracing point, but leaves us at liberty to impose upon that motion any arbitrary condition. This may be done by restricting the tracing point to move in a given line, or by assuming a relation between x_P and y_P indicated in general by the functional equation $F(x_P, y_P) = 0$; and, for each line thus assumed we have a distinct system of teeth. Now, the solution of the purely geometrical problem contains nothing to shew why one orbit should be preferred before another for the path of the tracer; therefore we must look to the mechanical desiderata for some guide in the selection. To prepare ourselves for engaging in this choice, let us examine the phases which appear when the simplest of all, viz. the straight line, is assumed for the path of the tracing point.

A B, figure 2, Plate I., being the line of centres, Q the pitch point, and P P′ the proposed line, let us endeavour to follow the course of the point of contact. Draw Q q to represent an element of the motion of the pitch lines, or their velocity, and having traced the ordinate P R, cut off P r equal to and in the same direction with Q q, join Q P and draw through r a line perpendicular to Q P, cutting the given line in p, then is p the proximate position of the tracing point, and the ratio of Q q to P p is equal to the ratio of the velocity of the pitch lines to the velocity of the tracer.

In our figure P is represented as below the line of centres, and P P′ is shewn as cutting that line between Q and R at the point C. Now, the motion of the pitch lines being supposed to be upwards, as long as P is below A B, the distance Q P must decrease, and this is compatible with a progress along P C. But as the point P approaches to C, the angle P Q B, and in consequence P r p which is equal to it, decreases to become zero, so that the velocity of the tracing point must also decrease as it nears C, there becoming zero. It may be then that the tracer can never reach C. In order to compute the time expended in passing over the distance P C, let us put Q C $= c$, and the angle C P R $= i$, then the equation of the straight line P P′ becomes

$$c - y_P = x_P \cdot \tan i \text{ . (10)}$$

Eliminating, by help of the differential of this equation, y_P and ∂y_P, from equation (9. a), we obtain

$$\left\{ \frac{c}{x_P} \tan i - \sec i^2 \right\} \partial x_P = \partial q,$$

which again, on being integrated, becomes

$$c \tan i \cdot \text{nep log } x - x \sec i^2 + A = q, \text{ (11)}$$

where the constant adjunct A must be so determined as to make the beginning of the motion count from some prescribed epoch, say from the instant when the tracer occupied the position P. Thus we have

$$- c \text{ nep log } \frac{x}{x_P} \cdot \tan i + (x - x_P) \sec i^2 = q. \text{ (12)}$$

If now we wish to ascertain what length of the pitch lines must run out during the motion of the tracing point from P to C, we must here put $x = 0$. But the logarithm of zero is infinitely negative, and therefore the tracing point which has left P can never reach C, so long as the term involving log. x is found in the equation. The tracing point then cannot describe continuously any straight line which crosses the line of centres, unless either c or tan i be zero.

In the first place, if c be zero, that is, if we suppose the inclined line to pass through Q, we have

$$x - x_P = Q \cos i^2; \text{ (13 . } a\text{)}$$

now the velocity of the tracer in its path, multiplied by cos. i, gives its velocity estimated in the direction x; hence the tracer must describe an oblique line passing through the pitch point with a velocity less than the velocity of the pitch line, in the ratio of the cosine of the angle of obliquity to radius.

In the second place, if i be zero, that is, if the path of the tracing point be perpendicular to the line of centres, we have

$$x - x_P = q; \text{ (13 . } b\text{)}$$

so that if P move in a straight line perpendicular to A B, its velocity must be uniform, and equal to the velocity of the pitch lines.

The only case of rectilineal motion not comprehended in equation (10) is that of a line parallel to A B. Returning to equation (9), and supposing x_P to be constant, we have, after integrating,

$$y^2 = 2\,x_P\,(q - A)\;.\;(13\,.\,c)$$

Here if x_P be negative, that is, if the line lie below A B, the value of y is possible from $q = -\infty$ to $q = A$, and has two equal values with opposite signs representing two points, which, coming from an infinite distance, simultaneously approach and disappear on the line Q X. If the path lie above A B, that is, if x_P be positive, the value of y is again double, and is possible from $q = A$ to $q = +\infty$. In this case two tracing points start together into existence on the axis of X, and separate the one to the right hand, the other to the left.

Having drawn, figure 3, Plate II., the line C Q D perpendicular to A B, it results, from the above investigation, that the point of contact can cross A B obliquely only at Q, every where else it must cross A B at right angles: also it can never cross the line C D at right angles.

WHATEVER may be the path of the tracing point, let us suppose the motion to be continued until the disc A have made a complete revolution; then is it evident that, unless the tracing point (or points, for there may be more than one), again occupy their original positions, there will be traced upon the disc two distinct contours, or more if the motion be continued during several revolutions. Hence it follows that, for all mechanical purposes, the original positions of the tracing point must be re-occupied at each entire revolution of the disc A. Not only so, the orbit must be retraced at each revolution of the disc B, and, consequently, at each repetition of the difference between the times of the two revolutions. So that, having found the common measure of the circumferences of the two pitch lines, the orbit of the tracing point must be repeated each time that the length of this common measure passes the point Q. Hence, if wheels be intended to revolve repeatedly, we have this indispensable condition, that the radii of their pitch lines must be commensurable, and that the distances Q A, Q A′, Q B, Q B′, etc. must all be multiples of one common measure, while the outline of each disc must consist of as many equal parts as its radius contains of this common measure.

Again, since, for general mechanical purposes, it is requisite that the wheels whose centres are on the side Q B, be exact counterparts of those whose centres are on the side A B, it follows that the orbit must lie symmetrically on each side of the line C Q D. Also, in order that the wheels may be susceptible of reversion face for face, the orbit must lie symmetrically above and below the line A Q B. Hence, altogether, the orbit of the tracing point must consist of four equal parts symmetrically disposed in the four right angles around Q.

First, supposing that there is only one point of contact, let us try to discover the general character of the simplest orbit which it can describe continuously.

If the pitch lines move in the direction C Q D, and if the tracing point begin its motion at some point *e* in the quadrant A Q D, its distance from Q must increase, and must continue to increase so long as P is above the line A Q B. Imagine it to describe some curve *e f g* crossing obliquely into the quadrant B Q D at the point *f*, then the symmetry of the arrangement requires that, at some subsequent period, it shall describe the similar arc *h f i* as also the symmetrically placed arcs *k l m* and *n l p* below the line A B. In order that the arrangement be regular, the extremities *h p*, *g k*, *e m*, *n i*, must be connected, or else *g h*, *e i*, *p k*, *m n*; but in neither way can there be a single continuous curve. The same argument would hold on supposing any even number of oblique crossings of the line C D; and we have seen that no perpendicular crossing of C D is compatible with the continuous motion of a single point of contact; therefore we are forced to look, for the solution of our problem, to some curve crossing C D at an odd number of points. Now there can only be an odd number of such points when Q is one of them, and hence we conclude that, if the tracing point move in a single continuous curve, it must cross twice through the pitch point Q.

The simplest of all such curves would evidently be that which has only one crossing; let us inquire if it be possible.

Starting from the pitch point, let P enter the quadrant A Q D in the direction Q *e*, figure 4, and, instead of turning to enter the quadrant B Q D, which would occasion at least three crossings, let it bend its course down to *f* on Q A. This path Q *e f* must be such that, while describing it, the tracing point may always recede from the origin Q, the velocity with which it recedes being proportional to the cosine of the angle D Q P. Hence in approaching the line Q A, while the angle D Q P approaches to be a right angle, the velocity of the recession must approach to zero, and therefore Q P must reach its maximum value at *f*. Crossing the line A B perpendicularly, the tracing point now enters upon the quadrant A Q C; here, the cosine of the obtuse angle D Q P being negative, the distance Q P must decrease. This approach of the tracing point to Q is clearly compatible with its passage along the curve *f g* Q symmetric to *f e* Q. Continuing its course, the tracer enters the quadrant B Q D, in which it describes the arc Q *h i*, and then it completes its orbit by tracing *i k* Q in the quadrant B Q C. Thus, during the passage of each tooth, the point of contact must describe the orbit Q *e f g* Q *h i k* Q. The simplest algebraical equation which can represent such a curve has the form

$$n x^2 = (Y + y)\, y^2\, (Y - y),$$

in which Y is put for *f* Q or Q *i*, and where *n* is some yet undetermined coefficient, whose value depends on the angle at which the curve crosses through Q.

The next degree of complexity which the continuous orbit of the tracing point can have, is obtained by taking three crossings, as at l, Q, f, figure 5, Plate III. Let $e\,f\,g$, $h\,f\,i$, $k\,l\,m$, $n\,l\,p$, t Q v, and r Q s, be six arcs of the curve near these crossings, then, if the path Q s run into $e\,f$, and if we make the symmetric junctions, the two parts $n\,l\,p\,r$ Q $s\,e\,f\,g$, and $k\,l\,m\,t$ Q $v\,h\,f\,i$, will result. In order to unite these, let the branch $e\,f\,g$ gradually bend down the cross A B perpendicularly at w, Q w being the minimum value of Q P, then will the symmetric junctions $w\,k$, $n\,z$, $z\,i$, be compatible with the continuous circulation of the tracing point in the orbit Q $s\,e\,f\,g\,w\,k\,l\,m\,t$ Q $v\,h\,f\,i\,z\,n\,l\,p\,r$ Q.

An orbit of this form would have some such equation as

$$x^2\,(\mathrm{X}^2 - x^2) - n^2\,y^2\,(\mathrm{Y}^2 - y^2) + x^2\,y^2\,(\alpha\,x^2 + \beta\,y^2 + \gamma) = 0,$$

in which the coefficients n, α, β, γ, determine the particular phases of the curve which has Q $f =$ X, Q $w =$ Y.

As our object is to examine the simpler forms, it is unnecessary to proceed to the case of five crossings, which would evidently lead to much greater intricacy: let us rather return to the case of a single crossing.

In an *Analysis of the Vibration of Straight Wires*, published in the Edinburgh New Philosophical Journal for April 1832, I have shewn that the curve, whose equation is $n^2\,x^2 = y^2\,(\mathrm{Y}^2 - y^2)$, is produced by the vibration of a wire whose maximum and minimum rigidities are in the ratio of four to one. The equation therein given, viz.

$$y = \mathrm{Y}\sin\ \theta;\ x = \mathrm{X}\sin\ 2\,\theta;\ (15)$$

will best suit our present purpose, on account of the facility which it affords in the calculations. In this equation X is the half breadth, and Y the half length of the knot, θ being a variable angle whose relation to the motion of the pitch line remains to be discovered. Somewhat analogous to the ellipse, this curve preserves always the same general character, although its form vary according to the ratio of X to Y.

In order that a single tracing point may describe continuously the whole curve, it is necessary that Q i be the maximum value of Q P: now we have $\mathrm{Q\,P}^2 = x^2 + y^2$, or

$$\mathrm{Q\,P}^2 = \mathrm{Y}^2\sin\ \theta^2 + \mathrm{X}^2\sin\ 2\,\theta^2.$$

Equating the differential coefficient of this to zero, we have $\mathrm{Q\,P}^2$ maximum or minimum, when

$$\sin 2\theta' \{Y^2 + 4X^2 \cos 2\theta'\} = 0.$$

This equation may be satisfied in two ways; thus we may either have sin. $2\theta' = 0$, or $Y + 4X^2 \cos 2\theta' = 0$. The first of these solutions gives $\theta = 0$, $\theta = 90°$, $\theta = 180°$, etc., which correspond to $x = 0$, $y = 0$; $x = 0$, $y = + Y$; etc.; it merely shews that QP^2 is minimum when P is at Q, and that it may be maximum or minimum when P is at i or at f; and this independently of the ratio X to Y. It is in the second solution that we must seek for information; from it we obtain

$$\cos 2\theta' = - \frac{Y^2}{4X^2},$$

which has a possible root whenever Y is less than 2 X. In all cases in which 2 X is greater than Y, there are four points in the curve whose distance from Q is maximum and greater than Q i; such curves cannot be traced continuously by a single point of contact. But when 2 X is less than Y, the fraction is greater than unit, and cannot be the cosine of any angle; therefore all such hour-glass curves as have 2 X less than Y can be described continuously by a single point. The limit of possibility is just when 2 X is equal to Y, and then it can easily be shewn that the circle which osculates the curve at the vertex i has its centre at the point Q.

Assuming some curve of this class for the path of the tracing point, let us investigate the law according to which the motion must take place. For this purpose we have to differentiate the equation of the curve (15), and to insert the values of ∂x and ∂y so found in the general equation of condition (9). In this way, after dividing by sin 2θ, we obtain

$$\{Y^2 + 4X^2 \cos 2\theta\} \partial\theta = 2X \partial q; \qquad (16 . a),$$

which integrated, on the supposition that q and θ start from zero together, gives

$$Y^2 \theta + 2X^2 \sin 2\theta = 2Xq; \qquad (16 . b).$$

If l denote the length of the orbit $\partial l^2 = \partial x^2 + \partial y^2$, whence

$$\partial l = \frac{2X \sqrt{\{Y^2 \cos \theta^2 + 4X^2 \cos 2\theta^2\}}}{Y^2 + 4X^2 \cos 2\theta} \partial q. \quad (16 . c).$$

In the equation (16 . b) which determines the relation between θ and q, the value of q is given explicitly in terms of θ, so that we can readily calculate at what instant the tracing point must reach any given position: but the converse problem, to find where the tracing point ought to be at any prescribed instant, is by no means so easy; it involves indeed the resolution of a transcendental equation. Fortunately, means can be found for reducing, within tolerable limits, the labour required for tabulating the results; we shall explain these as they present themselves.

When θ has passed through all values from $\theta = 0$ to $\theta = 2\pi$, the entire orbit of the tracing point must have been traversed once, and there must have been traced, upon the revolving disc, the complete outline of one tooth. Hence the distance from tooth to tooth, measured along the pitch line, must be the value of q corresponding to $\theta = 2\pi$; if Q be that distance, we have

$$2\pi Y^2 = 2XQ. \qquad . \qquad (17).$$

Putting R for the radius of a wheel of one tooth, so that nR may be the radius of a wheel of n teeth, we have $Q = 2\pi R$, and

$$Y^2 = 2XR \text{ or } 2X : Y :: Y : R, \qquad . \quad (18),$$

that is, *the radius of a wheel of one tooth is a third proportional to the whole breadth of the curve and its half length.*

When the number of teeth increases to become infinite, the wheel merges in the straight rack. Observing that $x = X \sin 2\theta$, we obtain from equations (16 . *b*) and (18)

$$\theta = \frac{q - x}{R}; \quad y = Y \sin \frac{q - x}{R}; \qquad . \qquad (19),$$

from which we can readily deduce the form of the tooth on a straight rack. In order to obtain this form, we must refer its position to a system of co-ordinates fixed in reference to the rack, and whose origin, therefore, moves in the direction X with the velocity of the pitch lines; hence we have $y' = y$, $x' = x - q$, wherefore

$$y' = -Y \sin \frac{x'}{R},$$

which is at once recognised as the equation of a curve of sines of which the half height is Y and the entire base $2\pi R$, R being the radius of the generating circle.

We might almost have anticipated this conclusion, for the teeth of a rack must evidently be uniformly disposed along the pitch line C Q D, and, if they have one continued contour satisfying the condition of reversion, that contour must oscillate symmetrically from side to side. Now the curve of sines is the simplest analytic curve which possesses this property, and thus we might have assumed it as the type of our system, and deduced, from its properties, the law of the motion of the point of contact.

In order to adapt our formulæ to trigonometric calculations, let us measure the motion of the pitch lines by the angular motion of a wheel of one tooth of which the pitch-radius is R. This angular motion being denoted by α, we have $q = R\,\alpha$, whence

$$\frac{X}{R} \sin 2\,\theta = \alpha - \theta. \qquad . \qquad (20).$$

THAT there may be only one point of contact, and that that point may describe the orbit continuously, we have already seen that Y must not be less than 2 X; but without, for the moment, attending to this limit, we proceed to investigate, in another way, the law of the number of roots of equation (20).

Let O A B C D E F, etc. be a curve of sines of which the base O D is $2\,\pi$, and of which the half altitude is unit. Make figure 6, Plate IV., an angle B O T, having for its tangent the ratio $R \div 2\,X$, or $R^2 \div Y^2$; measure off $O\,M = 2\,\theta$, and draw the line M N P perpendicular to the axis of the curve, then, from its genesis, $P\,M = \sin 2\,\theta$, whence $M\,N = \frac{R}{X}\,\theta$, wherefore

$$P\,N = \sin 2\,\theta + \frac{R}{X}\,\theta = \frac{R}{X}\,\alpha.$$

Through P draw R P S parallel to O B, meeting the ordinate through O in R, and the axis in S. Then O R : O S : : R : 2 X, but O R is equal to P N, therefore O S is equal to $2\,\alpha$.

In order to find what value of α corresponds to any proposed value of θ, we must measure off O M equal to $2\,\theta$, raise the ordinate M P, and through P draw P S parallel to O T, then does O S represent the value of $2\,\alpha$. Conversely, in order to find that value of θ which accompanies a given value of α, we measure off O S equal to $2\,\alpha$, draw through S a parallel to O T, cutting the curve in P, and let fall the ordinate P M, then is O M equal to $2\,\theta$.

If the line R S cut the curve only once, there will be a single point of contact; but, if the parallel to O T cut the curve several times, there will be several values of θ corresponding to one value of α, and therefore there will be several contemporaneous contacts. Now the greatest inclination of the normal curve of sines to its axis is $\frac{\pi}{4}$, and therefore, if the angle B O T be not less than half a right angle, that is, if Y be not less than 2 X, the parallel to O T can never cut the curve oftener than once, and there can never be more than one contact. But if B O T be less than half a right angle, S P may cut the curve 3, 5, 7, or any odd number of times; a contact being regarded as the mergence of two intersections, that is, as two intersections in the act of appearing or disappearing. Hence, in all wheelwork deduced from the hour-glass curve, the points of contact must appear and disappear in pairs, and, except at the instant of coalescence, their number is always odd.

When the angle B O T is somewhat less than half a right angle, R S, parallel to O T, may sometimes cut the curve of sines in three, sometimes only in one point, according to the particular position of the point S. But, by sufficiently reducing the angle, there may come to be always three intersections: this will be when O T touches the curve of sines near the summit C.

If the obliquity be still farther diminished, there will be sometimes *three* and sometimes *five* intersections; until, when O T is brought to touch the curve near its vertex G, there will be always five intersections, neither more nor less.

Thus, by computing the inclinations which O T must have in order to touch the curve near its successive vertices, we shall be enabled to ascertain what form must be given to the hour-glass curve in order that the wheels obtained from it may always have a proposed odd number of contacts.

Let O U touch the curve near the summit C, and draw the ordinates U V, C W, then, if the arc W V be represented by ψ, we have V U = cos ψ, but, from the nature of contact, ∂ . U V : ∂ W V : : U V : V O, whence

$$\cot \psi = \frac{3}{2}\pi - \psi ;$$

and, in general, if n be the number of contacts,

$$\cot \psi = \frac{n}{2}\pi - \psi \qquad . \qquad (21)$$

$$\frac{2\,X}{R} = \operatorname{cse} \psi ; \frac{Y}{R} = \sqrt{\operatorname{cse} \psi} \qquad . \quad (22).$$

In all cases, except when $n = 1$, ψ is a small angle, and the variation of its cotangent is rapid; so that if we place, in the first member of equation (21), some assumed value for ψ, we shall obtain, from the second member, a closer approximation, by help of which we can obtain a still closer, the convergence being very rapid. Also adding tan ψ to each side, and observing that tan. ψ + cot ψ = 2 cse 2 ψ, we have

$$\operatorname{cse} 2\,\psi = \frac{n}{4}\pi + \frac{\tan \psi - \psi}{2} ; \quad (21 . b)$$

now tan ψ — ψ is extremely small, and therefore for a first approximation we may take

$$\operatorname{cse} 2\,\psi = \frac{n}{4}\pi .$$

When begun in this way, the rapidity of the calculation leaves nothing to be desired. The results are exhibited in the following table:

n	ψ	Log cot ψ
3	12°..32′..47″.77	0.652 5760
5	7 ..22 ..32 .43	0.887 9127
7	5 ..14 ..23 .46	1.037 5906
9	4 ..03 ..59 .24	1.148 1766
11	3 ..19 ..24 .25	1.236 0522
13	2 ..48 ..37 .142	1.309 0188
15	2 ..26 ..04 .688	1.371 4272
17	2 ..08 ..51 .484	1.425 9588
19	1 ..55 ..16 .353	1.474 3852
21	1 ..44 ..16 .800	1.517 9396
23	1 ..35 ..12 .147	1.557 5147

Assuming, as we shall do throughout the subsequent part of the work, unit for the pitch radius of a wheel of one tooth, the values of X and Y set down in Table I. of the Appendix have been at once obtained.

LET us now begin with the simplest case, and examine the form of the deepest tooth which shall never have more than one point of contact. In this case $Y = 2\,X = R = 1$, wherefore

$$\sin 2\,\theta = 2\,\alpha - 2\,\theta; \qquad . \qquad (23)$$

$$x = \tfrac{1}{2} \sin 2\,\theta = \alpha - \theta;\ y = \sin \theta. \qquad (24)$$

Dividing the entire revolution of the wheel of one tooth into 40 equal parts, each containing 10 centesimal or 9 ancient degrees, and making α successively 0°, 9°, 18°, etc.; let us endeavour to compute the successive values of θ, x, and y.

The path of the tracing point near Q is almost straight, and is inclined at half a right angle to the line Q D, wherefore (13 a), the rate of increase of x there must be half the velocity of the pitch lines, therefore for $\alpha = 9°$ we shall have x nearly $\frac{1}{2}\alpha$, whence θ must be about 4° 30′. From this approximate value of θ a closer approximation may be obtained by inserting the previous value in the second member of the equation $2\,\theta = 2\,\alpha - \sin 2\,\theta$; and this process can be continued until the result be sufficiently exact. The ultimate result is $2\,\theta = 9° \,..\, 01' \,..\, 07''$, from which we easily obtain the values of x and y. For $\alpha = 18°$, it is evident that θ must be somewhat more than 9°, and as the difference between the sine of a small arc and the arc itself is nearly proportional to the cube of the arc, we may take about eight times the preceding excess, and assume as our first approximation $2\,\theta = 18° \,..\, 09'$, from which we readily obtain the true value, 18° .. 09′ .. 04″. In this way the values of x and y corresponding to each value of α from 0° to 90° have been computed; the values for the other quadrants being obtained by a mere change of signs. These values are entered in Table II. of the Appendix.

By help of this table of the co-ordinates of P, we easily trace the orbit represented in figure 7, Plate III., where the numbers 1, 2, 3, up to 40, shew the positions of the point of contact at each of the 40 intervals into which the passage of a tooth is divided. Since the curve, at its summit, is osculated by a circle described from Q, the velocity of the tracing point near the divisions 10 and 30 is infinite, whence the great separation of the points 9, 10, 11; 29, 30, 31.

By means of this diagram we can easily obtain the form of a wheel of any proposed number of teeth. Observing that Q B is the radius of a wheel of one tooth, we must measure off, for a wheel of n teeth, a distance along Q B equal to n times Q B, in order to obtain the position of the centre. Next we divide a blank disc into n parts, one for each tooth, and each of these again into forty parts, and pin the centre of this disc at the point just found; the outer edge of the disc ought to pass a little beyond A. The paper on which the orbit is traced being secured to the board at its farther edge, is now raised to allow the disc to pass below it, and the apparatus for delineation is prepared. Bringing the division 0 to the reading mark, we pierce with a fine needle through the point 0 on the orbit: turning the disc round to division 1, we mark through the point 1, and so on till the disc have made a complete revolution; then is the entire outline of the wheel traced out.

Those who have felt any difficulty in following our investigations would do well to practise this method of projection: all the apparatus required is a flat board, on which the centre of a circular disc of stiff drawing paper may be kept in its place by a needle, while the orbit is traced on a card, whose farther edge is secured by glue or otherwise to the board. Care must be taken to have the needle exactly at the proper distance from Q, and truly in the continuation of the line A Q B.

When n is infinite the wheel merges into a straight rack, the form of which is easily obtained by causing a blank slip of paper to move in the direction C Q D by steps of $\frac{\pi}{20}$ at a time. In figure 8, Plate V., the two extremes of the series are shewn; Q F G H being the pitch line of a wheel of one tooth, whose centre is at C, and D Q E being the pitch line of the straight rack.

It is obvious at a glance that such a form of tooth, although it satisfy perfectly the geometrical condition of contact, is utterly inadmissible in practice. Independently of the obliquity of the action, which itself is a fatal objection, there is this other, that during part of the motion, one wheel must be the driver, and during the other part, the other wheel. We are thus forced to seek, in an augmented number of contacts, for those conditions which are indispensable in practice.

PROCEEDING to the case of three contacts, we have $\psi = 12^\circ .. 32' .. 47''.77$, whence $X = 2.301\,670$, $Y = 2.145\,540$; and, constructing an hour-glass curve with these dimensions, we have the form shewn in figure 9, Plate VI. A slight inspection of this figure will serve to shew that the tracing point cannot describe the curve continuously; for, so long as it is below the line of centres A Q B, it must be approaching to Q. Now a circle described from Q, with the radius Q K, falls within the curve; and therefore, if the tracing point were to descend from K towards F, as it would need to do in describing the curve continuously, its distance from Q would be increasing. The maximum distance from Q is at some point F, and a circle described with the radius Q F, would touch the curve at the four points F, G, H, and I. Let the tracing point begin its motion at F; it may ascend along the branch F K, to reach its minimum distance from Q at K, and, passing into the other quadrant, may continue to ascend until it reach G; but there it must necessarily disappear, since it cannot return along G Q. Or, beginning at F, the tracing point may ascend along the branch F Q, and continue its progress in the opposite quadrant, to disappear at I. Similarly P, having made its appearance at H, may describe the branches H L I and H Q G, to disappear at I or at G: and this must be the general character of the motion in any hour-glass curve whose half length Y is less than its breadth 2 X.

Those contacts which occur along the lines F K G and H L I, are, comparatively, of little practical value; they serve to complete the continuity of the outline, and to prevent the wheels from locking: those which occur on the lines F Q I, H Q G, are to be considered as the true working contacts. To the former we shall assign the name *plerotic* or supplementary, to the latter *pragmatic* or effective. These classes of contacts are distinguished by the obliquity of the action. Thus the extreme obliquity of the pragmatic actions is determined by the angle D Q F, and this obliquity decreases to attain its minimum when P arrives at Q. But the obliquity of the plerotic contact increases to become an entire right angle, and the action, therefore, absolutely ineffective, when the tracing point reaches K. The minimum obliquity of the pragmatic action is the angle at which the curve cuts the line D Q E: denoting it by i, we have

$$\cot\ i = \frac{2\,X}{Y} = \sqrt{\mathrm{cse}\ \psi}. \quad (25)$$

In order to obtain the greatest obliquity of the pragmatic action we must ascertain the position of the point G, where Q P is a maximum. We have already obtained for the point G,

$$\cos\ 2\,\theta' = -\frac{Y^2}{4\,X^2} = -\frac{1}{2\,X} = -\sin\ \psi;$$

$$\text{wherefore } 2\,\theta' = \frac{\pi}{2} + \psi. \quad . \quad (26).$$

Now the angle P Q E has for its tangent the expression $\frac{y}{x}$, whence

$$\tan \text{E Q P} = \frac{\text{Y} \sin \theta}{\text{X} \sin 2\theta} = \frac{1}{\text{Y} \cos \theta};$$

so that if ω be the angle E Q G of greatest obliquity, we must have

$$\cos \omega = \tan \left(\frac{\pi}{4} - \frac{\psi}{2}\right) \quad . \quad (27).$$

The value of $2\theta'$, contained in equation (26), shews that the appearance and disappearance of the points of contact is simultaneous with the appearance and disappearance of the intersections of the line R P S of figure 6 with the curve of sines.

From these equations we at once obtain $i = 24° \,..\, 59' \,..\, 22''$, and $\omega = 36° \,..\, 41' \,..\, 11''$, between which limits the obliquity of the action of wheels belonging to this system of three contacts is always included.

In order to compute the motion of the point of contact, we must insert the values of X and Y above obtained, in the general equations

$$\text{X} \sin 2\theta = \alpha - \theta = x; \ \text{Y} \sin \theta = y.$$

Dividing, as before, the revolution of a wheel of one tooth into forty equal parts, we have to compute the values of θ, x, and y, corresponding to $\alpha = 9°$, $\alpha = 18°$, etc. Now, near to the point Q, the motion is almost rectilineal, and we have $x \doteq \alpha \cos i^2$, whence $\theta \doteq 1° \,..\, 36' \,..\, 41''$; so that we may assume, as a first approximation, $2\theta \doteq 3° \,..\, 13' \,..\, 20'$. Inserting this in the second member of the equation, $2\,\text{X} \sin 2\theta = 2\alpha - 2\theta$, we obtain a nearer value; and continuing this process until there be no appreciable error, we obtain $\theta = 1° \,..\, 36' \,..\, 24''.76$.

Having thus found the value of θ for $\alpha = 9°$, we can assume somewhat less than the double of that for the first approximation to the value of α for $\theta = 18°$, which has then to be corrected in the same way. Tabulating the results, and attending to the successive differences, we easily guide our first suppositions so as to render the subsequent corrections very short. With every appliance the labour of these operations is very great; particularly for those contacts which take place near the point G: it may be considerably reduced by attending to the tabular differences, or by using the equation

$$\text{correction of } 2\theta = \frac{-\text{error}}{1 + 2\,\text{X} \cos 2\theta}.$$

Having continued this process to $\alpha = 90°$, and obtained $2\theta = 33° \,..\, 41'$

.. $29''.3$, we have traced one quarter of the motion of one of the points of contact. Thence we may pass to the second point, by subtracting the above result from 360°, which gives for $\alpha = 90^\circ$, also $2\,\theta = 326 \,..\, 18' \,..\, 30''.7$, satisfying equation (20). From this we return through $\alpha = 81^\circ$, $\alpha = 72^\circ$, etc., to $\alpha = 0$, when $2\,\theta$ is found to be $257^\circ \,..\, 27' \,..\, 12''.2$; and thence again we repass through $\alpha = 9^\circ$, $\alpha = 18^\circ$, etc., to $\alpha = 90$, at which the third value of $2\,\theta$ comes out 180°.

From these values of $2\,\theta$ those of x and y can readily be obtained, and we are then enabled to indicate the motion of each of the three tracing points during one quarter of the passage of a tooth; the positions for the other three quarters being given by a mere change of signs. In this way the results given in Table III. of the Appendix have been obtained. By help of these co-ordinates, the positions of the three tracing points at each of the forty intervals have been marked off, as shewn in figure 9, Plate VI.

At the beginning of the motion, when $\alpha = 0$, one tracing point is at Q, about to move towards G; the second is at I, just disappearing; and the third, having appeared at F, is ready to separate into two, one of which will pass along F K G, the other along F Q. The simultaneous motions of the three tracing points are clearly indicated by the progression of the numbers. The contacts which appeared together at F, reach the points G and Q simultaneously, and at that instant the contact which left Q also reaches G: at this epoch, which corresponds to $\alpha = 180^\circ$, two contacts coalesce and are about to disappear at G, while two others are making their appearance at H. Similarly these new contacts pass along the branches H I and H Q, and thus, during the transit of one tooth, trace out the whole of the orbit.

In figure 9, the distance $Q\,C'$ is unit or R, the radius of a wheel of one tooth. Since this wheel has its centre within Q K, the half depth of the tooth, it cannot have a mechanical existence: however, if we place at C' the centre of a moveable disc, whose revolution is divided into forty equal parts, and if, turning it round one division at a time, we, at each instant, mark on it the corresponding positions of the three tracing points, we shall obtain the curve delineated in figure 10, Plate VII., which perfectly satisfies the geometrical condition of contact.

If, having measured off $Q\,C''$ equal to $2\,R$, we make C'' the centre of a disc whose revolution is divided into eighty parts, and proceed in the same way, we obtain figure 11, Plate VII., which exhibits the form of a wheel of two teeth having always three points in contact. Since $Y = Q\,K = 2.145$, this point C'' must also be within the half depth of the tooth, and, from this circumstance, we might at once have recognised the mechanical impossibility of such a wheel, albeit the geometrical conditions be rigidly fulfilled.

On taking $Q\,C'''$ equal to $3\,R$, we obtain the centre of a wheel of three teeth: dividing the revolution of a disc whose centre is at C''' into three times forty equal parts, and proceeding as before, we have the form represented in figure 12, Plate VIII. Here, although the bottom of the tooth do not reach to the centre of the wheel, the overlapping of the lines shews clearly that the mechanical construction of such a wheel is impossible.

Now, if we continue to remove the centre of the disc still farther from Q, we shall at length have a form closely approaching to the curve of sines, and therefore free from knots; so that the question arises, *at what distance must* C *be placed in order that the outline of the tooth may not return upon itself?* The resolution of this question will give the limit of mechanical possibility.

Let us suppose that the centre of the disc is removed to C, figure 13, Plate VIII., somewhere beyond the vertex A of the curve; then we observe, that if the paths of the tracing points at A and B were straight lines perpendicular to the axis, the velocity of the tracing point there would be (13, *b*) exactly equal to that of the pitch lines: but the linear velocity of the disc, measured at the distance C Q, is also that of the pitch lines, wherefore the linear velocity of the disc at the distance C A would be less than that of the tracing point at A in the ratio C A : C Q, so that there would be no overlapping at the bottom of the tooth; while the linear velocity of the disc at the distance C B being greater than that of the tracing point in the ratio of C B : C Q, the trace would be made backwards upon the disc, and therefore there would always be an overlapping or pennon at the point of the tooth. However, the path of the tracing point at A and B is not straight, but concave towards Q, therefore the velocity there must be greater than that of the pitch lines, and it is possible to find a point C such that the ratio C B : C Q is less than the ratio between the two velocities, in which case there would be no overlapping at the point of the tooth.

Now, at the bottom of the tooth, $\theta = 90°$, wherefore (16, *c*) the velocity of the tracing point there is given by the equation

$$\partial l = \frac{Y^2}{Y^2 - 1} \partial \alpha.$$

If, then, we assume a point M, such that Q M : B M : : $Y^2 - 1 : Y^2$, or, since $QB = Y$, if we take $BM = Y^3$, the linear velocity of the disc whose centre is at M, measured at the distance M B, will be just equal to the velocity of the tracing point there. On any disc whose centre is beyond M, there will be no overlapping at the point of the tooth; but, on all discs whose centres are between Q and M, a doubling must be formed on the curve, and thus M is properly the *metacentre*, indicating by its position the minimum wheel which, having the full number of contacts, can exist mechanically. It is worthy of remark, that the distance of the metacentre from the summit of the tooth is the fourth of the continued proportionals

$$R : Y :: Y : 2X :: 2X : BM,$$

and that, therefore, R being reckoned as unit,

$$QM = Y^3 - Y \quad . \qquad (28).$$

We may also notice that M is the centre of the circle which osculates the hour-glass curve at the point B, and that, therefore, the circle described from any point C beyond M with the radius C B, includes within it the entire orbit.

But the summit of the tooth is not the only region at which a replication of the curve may be found: the preceding figures shew pennons also on the shoulders. Now the tracing point, while describing the curve H Q G, ought to be continually approaching to the centre C, for if it approach to reach a minimum distance, then recede and again approach, the outline of the tooth will be knotted or undulated.

Assume any point p in the curve Q G, figure 13, and draw the normal $p\,n$. The radius of curvature at Q is infinite, and decreases as the point p moves along the curve: suppose that it has not yet become less than $p\,n$, or that the centre of curvature for the point p is beyond n: then if the centre of a disc were placed at n, the tracing point, in passing from Q to p, would approach and reach its minimum distance at p: but in proceeding beyond p, it would again shew a tendency to recede, and there would be undulations on the side of the tooth. The centre of the disc must be placed beyond n. Now as p is shifted along the line Q G, the distance Q n increases to attain a maximum value, and then decreases to become zero when p reaches G. Let P N shew the maximum value of Q n, then, from the nature of osculation, N must be the centre of curvature for the point P, and therefore if C were placed at N, the distance p C′ would decrease just to become stationary at P, and would again decrease until p reached G. N is thus the limit at which the replications disappear from the shoulders of the teeth.

We have in general

$$Q\,n = Y \sin\theta\,\{1 + Y^2 \cos 2\theta\};$$

equating the differential coefficient of this to zero, in order to ascertain when it is stationary,

$$0 = Y \cos\theta'\,\{1 + Y^2 - 6\,Y^2 \sin\theta'^2\}.$$

This equation may be satisfied in two ways: firstly, by putting $\cos\theta' = 0$, whence $QN = \pm Y \mp Y^3$, agreeing exactly with the value just obtained for Q M; secondly, by putting

$$0 = 1 + Y^2 - 6\,Y^2 \sin\theta'^2,$$

which gives

$$QN = \sqrt{\left\{\frac{2}{27}(\gamma^2 + 1)^3\right\}} \quad . \quad (29).$$

The point N thus determined is another metacentre, such that when Q C is less than Q N there is a replication or knot on each side of the tooth. Of the two metacentric distances Q M and Q N, Q M is, in all practical cases, the greater, for

$$27\,(QM^2 - QN^2) = (5\,Y^2 - 1)\,(5\,Y^2 - 1)\;(Y^2 - 2)\,;$$

this shews that $QM^2 - QN^2$ is negative for all values of Y^2 less than $\frac{1}{5}$, is just zero when $5\,Y^2 = 1$ (which is a double root of the equation $QM^2 - QN^2 = 0$), is again negative from $Y^2 = \frac{1}{5}$ to $Y^2 = 2$, when it a second time becomes zero, and is positive for all values of Y^2 greater than 2. Hence, in all practical cases, the intrication of the curve disappears from the sides of the tooth previous to its disappearance from their summits. The values of Q M and Q N for each number of contacts, are given under their proper heads in Table I. of the Appendix.

For the system of three contacts, the value of Q M is 7.731, so that all wheels of 8 teeth and upwards have simple contours and the complete number of contacts. Below the number 8 there are replications at the points of the teeth, and, since Q N = 3.610, as soon as we descend to three teeth, replications appear also on the shoulders. Although, mechanically speaking, the teeth of wheels whose radii are between Q M and Q N be defective, the defect is chiefly in plerotic contacts, so that the wheels may still be effective: but below Q N the most advantageous contacts disappear and the wheels are entirely useless. Between the two metacentric limits the teeth are pointed; beyond the limit Q M they are rounded; and, for this class of wheels, we may fix the distinction between *pinions* and *wheels* at Q M. In this way the system of three contacts has pinions of 4, 5, 6, and 7 leaves, and the smallest wheel has 8 teeth.

The extent of the replication of the pinion leaf varies with the distance Q C, and it may or may not exceed the region of plerotic contact; so long as it is confined within this region it cannot in the least impair the effectiveness of the wheel. Let us endeavour to discover where C must be placed, in order that the replication may just extend to the points H and I; we shall then have a third metacentric distance shewing the limit of complete pragmatic action.

When the centre of the disc is placed within Q M, the velocity of the tracing point near B is less than that of the disc, and the trace is backwards, but gradually the velocity of P increases to become equal to, and at last to exceed, that of the disc. The angular distance of P from a line fixed upon the wheel in the original direction C B, will thus increase to reach a maximum and then decrease to become zero: and thus, if I be the limit of the replication, the angle B C I must be just equal to that described by the disc during the passage of the tracing point from B to I. Now the value of α corresponding to the motion B I is, for three contacts 90°, and in general for n contacts is $(n-1)\frac{\pi}{4}$, so that the disc must turn through the angle $\frac{n-1}{Q\,O}\,\frac{\pi}{4}$, wherefore the value of Q O is to be determined from the extremely involved equation

$$\frac{I\,R}{R\,Q + Q\,O} = \tan\left(\frac{n-1}{Q\,O}\,\frac{\pi}{4}\right), \quad . \quad (30)$$

which can hardly be resolved otherwise than by trial. For all practical purposes, it is sufficient to know the two consecutive integer numbers between which Q O is placed: but that nothing may be awanting in the exactitude of our investigations, the values of Q O for the different numbers of contacts have been carefully computed and entered in Table I. of the Appendix. For three contacts Q O = 4.502, so that the pinion of four leaves is defective in pragmatic contacts.

In the system of three contacts, two pragmatic actions are accompanied by one plerotic. Of the pragmatic contacts, one is backwards and one forwards, so that there is always one point sustaining the pressure of the machinery.

It is essential to the secure action of any train of wheels, that more than one tooth should have hold at a given instant, lest the breaking or derangement of a single tooth occasion extensive damage. Also the greater the number of teeth that are in action at once, the more will the gradual wearing of the parts tend to equalize the divisions and to improve the action. Now, if we desire to have t teeth always in complete action, we must have t pragmatic contacts forwards and as many backwards, in all $2\,t$; these again must be accompanied by $2\,t - 1$ plerotic contacts, so that altogether there must be $4\,t - 1$ contacts in order that t teeth be always in action at once. Thus the first system which can fully satisfy the wants of the machine maker is that of seven contacts, and we might proceed directly to consider its phases. Nevertheless we shall, for the sake of continuity, develope the system of five contacts, in which sometimes two teeth are engaged, and sometimes only one.

On reference to Table I. we find that, in the system of five contacts, Q M = 18.950, so that all wheels of 19 teeth and upwards have their outlines complete; Q O = 10.779, therefore all pinions of 11 leaves and upwards have their pragmatic actions perfect; lastly, Q N = 7.092, wherefore pinions of 7 leaves and less are quite impracticable. In this class of wheels, the action, at its best, has an obliquity of 19° .. 43′, at its worst, of 28° .. 29′, and the pitch, or distance from tooth to tooth measured on the pitch line, is to the entire depth of the tooth as 3.1416 : 2.791, or almost exactly as 9 : 8.

Proceeding as before we obtain, for the forty intervals into which we have divided the interval between two teeth, the values of x and y registered in Table IV.; by help of which we readily construct figure 14, Plate IX., shewing the simultaneous positions of the five tracing points. Combining these positions with the subdivisions of a disc whose centre is at C', we obtain figure 15, Plate X., as the form of a wheel of one tooth belonging to this system. Figure 16, Plate XI., shews a straight rack engaged with a wheel of 20 teeth.

Augmenting the number of contacts by two, we obtain three plerotic and four pragmatic contacts, indicating two teeth always in complete action. In this system Q M = 32.925, so that the smallest wheel with its seven contacts complete must have 33 teeth. Below that number knots appear at the points of the teeth in the region of plerotic contact, and gradually extend till, on passing the limit Q O = 18.530, they trench upon the working contacts, so that a pinion of 19 leaves is the smallest which can have its action complete. When, however, the pressures are light, we may descend to the inner limit Q N = 11.242, that is, to the pinion of 12 leaves, since, although one contact be lost, the other still retains its hold. Yet pinions within the limit Q O are always to be avoided if possible. But when we descend below Q N to the pinion of 11 leaves, those very contacts, the obliquity of whose action is the least, are deficient at the same time with those near the point.

In this system the obliquity is included between the limits 16° .. 49′ and 24° .. 09′, and the base of the tooth (always measured on the pitch line) is to its entire depth as 3.1416 : 3.309, or as 75 : 79 nearly.

The values of x and y belonging to each of the seven tracing points have been computed and recorded in Table V. From these again the orbit represented in figure 17, Plate XII., has been projected, and thence figure 18, Plate XIII. shewing the rack engaged with a pinion of 19 leaves.

This system is the simplest which can be adopted for general use: wheels belonging to it possess the property of tending to wear into better action, since, if one tooth be forward of its true position it will touch alone, and, receiving a greater pressure, will be worn more speedily until the other contact come to relieve it. Should the numbers of the teeth in the two wheels be prime to each other, each interval of the one comes, in the course of many revolutions, to be applied to every interval of the other, and the wearing tends to render the division more and more exact.

In the system of nine contacts, five are pragmatic and four plerotic. Thus we have, during part of the motion, three acting contacts forwards, and during the other part only two; so that we may say that there are two teeth and a half engaged. For this system we have Q N = 15.972, so that the smallest pinion must have 16 leaves: again, Q O = 27.523, that is, the pragmatic action is incomplete until we reach 28 leaves, and it is not till we pass Q M = 49.200, and reach the wheel of 50 teeth, that the whole outline is expanded. The obliquity of the action is between 14° .. 55′ and 21° .. 20′, and the base is to the depth of the tooth as 3.1416 : 3.755, or as 41 : 49 nearly. Table VI. contains the values of x and y for each of the nine tracing points; while figure 19, Plate XIV., exhibits the form of the orbit, and the manner of the motions therein.

The system of nine contacts, though more advantageous in its action than that of seven, does not offer a resting point to the practical engineer, who naturally looks to an increase in the number of acting teeth as a source of greater security and uniformity in the working of the machinery. If, then, the system of seven contacts do not possess these qualities in a sufficient degree, he would prefer to step at once to the system of eleven contacts, which gives three teeth in complete action at once.

In the system of eleven contacts, six are pragmatic and five plerotic; the obliquity of the action lies between the limits 13° .. 32′ and 19° .. 20′; while the ratio of the base of the tooth to its height is 3.1416 : 4.1533, or almost exactly 59 : 78. Since Q N = 21.219, the smallest possible pinion is that of 22 leaves; while, since Q O = 37.608, the pinion of 38 leaves is the smallest which has its pragmatic actions complete; and as Q M = 67.490, the wheel of 68 teeth is the smallest which has the outline entirely expanded. Table VII. exhibits the values of x and y for each of the tracing points, and, by its help, the orbit shewn in figure 20, Plate XV., has been protracted.

It is hardly necessary to prosecute the investigation of the phases of a greater number of contacts; since we have already reached as great a depth of tooth as is desirable; and since the smallest pinions are, even with eleven contacts, too high to permit of the general application of the system. The principal dimensions of the orbits up to 23 contacts are given in Table I.

Having now obtained the exact motions of the tracing points, we can delineate the forms of the teeth by a simple geometrical process. Thus, the successive positions of the tracing points being indicated by minute holes pierced through a sheet of thin brass; we secure this plate over a revolving blank disc, taking care that the line L Q K pass exactly through the centre C, while Q C is the n^{th} multiple of R. Then, turning the disc round step by step, we mark upon it by inserting a fine needle through the proper apertures. This method requires apparatus of considerable magnitude, and an elaborately divided circular index. It is more convenient and much more exact to compute the co-ordinates of the odontoid, so as to be able to trace it out by help of the co-ordinate machine.

Assuming A as that value of α from which we are to count the motion; the position of the tracing point at any time α being given by the co-ordinates x and y, we must, in order to obtain the position which the mark of that point upon the disc occupied at the epoch A, turn the disc back through the angle corresponding to α — A. If n be the number of teeth, n also represents the pitch radius of the wheel, or the distance Q C of figure 21, and hence $\frac{\alpha - A}{n}$ is the angle through which the disc must be turned.

Suppose P, figure 21, Plate IX., to be the position of the tracing point at the instant α, make the angle P C Π equal to $\frac{\alpha - A}{n}$, and cut off C Π equal to C P, then Π is the position which the mark of the tracing point upon the moveable disc occupied at the epoch A. Put (x), (y) for the co-ordinates of a point in the odontoid, then we easily obtain

$$\left.\begin{aligned}(x) &= y \sin \frac{\alpha - A}{n} + x \cos \frac{\alpha - A}{n} - n \sin \frac{\alpha - A}{n},\\ (y) &= y \cos \frac{\alpha - A}{n} - x \sin \frac{\alpha - A}{n} + n \operatorname{ver} \frac{\alpha - A}{n},\end{aligned}\right\} \quad (31)$$

formulæ which are remarkably convenient for systematic calculation.

The last terms are the co-ordinates of the pitch line, and are common to all systems of odontoids; the other terms depend on that particular system which is adopted. The most convenient epochs are A = 90° or A = 270°, since the curve then lies symmetrically above and below the line Q C, allowing the extent of the calculations to be doubled by a mere change of sign in the x.

THAT system of engrainage of which we have been treating, is the simplest which, satisfying the very important requisite of permutability among the wheels, is susceptible of a complete analytic representation. So far it possesses some of the distinctive characters of optimism; yet we must not, on that account, conclude that, under every point of view, it presents the greatest recommendations. There may be other mechanical desiderata equally important with interchangeability: indeed, if we except the screwing and self-acting lathe, the dividing engine, and one or two other machines, in which the relative velocities have to be changed, it is enough, in the great majority of cases, that two wheels gear properly into each other, and then we may substitute for permutability any other desideratum, such as minimum friction, facility of formation, etc. Nay more, we have considered only one case of permutable wheels, and among the infinity of others there may be some which possess ulterior advantages.

For whatever purpose the wheels may be intended, it is natural that the manufacturer, in making his preliminary arrangements, should look for some system which may diminish, as far as possible, the number of his tools, and simplify, to the greatest degree, his apparatus; and thus we can hardly judge of any particular system until we have examined the general principles of the entomy of wheels.

All methods for cutting out the teeth of wheels may be classed under four heads.

In methods of the first class, the cutter is so shaped as to give at once the desired form, and the whole accuracy of the workmanship depends on the formation and adjustment of the finishing cutter.

The most common of this class of methods is to mount a revolving or rose-cutter upon an axis exactly perpendicular to the axis of the blank wheel, and to bring the cutter down upon the edge of the blank by means of a guide: by this operation the space between two teeth is cut out. The cutter is now raised, the blank turned through the angular distance between two teeth, and another cut is made: this proceeding, sufficiently repeated, notches out the whole periphery of the wheel. The rose-cutter, however, is a tool tedious and expensive in its original formation; during the process of hardening, it expands, and often expands unequally; when blunted, it is sharpened with great difficulty, almost with the certainty of having its form deranged. To facilitate the manufacture of these cutters, a custom has been introduced of sweeping out their sides by means of another rose-cutter, so that the outline of the side of the tooth may be an arc of a circle: this method has the rather inconvenient attribute of being unfit, under any possible combination of circumstances, to give a true result. Also, although in every possible system, the form of the tooth vary from wheel to wheel, the expensiveness of the rose-cutter leads many to use but one for all wheels of one pitch, or perhaps one for the pinions and one for the wheels. Hence arises much of the imperfection which we witness every day in the formation of teeth, and of which the infallible tokens are tremour and noise.

Another method of this class, much more economical, and, in many respects, more convenient, is to employ a modification of the bar-cutter, first introduced, I believe, by Mr Joseph Whitworth of Manchester. A series of steel bitts, each projecting a little beyond its antecedent, is adjusted on a sliding bar, and the last bitt is carefully made to the required shape. The bar is caused to slide in a direction parallel to the axis of the blank, so that each bitt may remove a portion of the metal, and prepare the way for its successor, until the last bitt finish the cut. In this arrangement the steel pieces receive their finished shape after having been hardened; they can easily be removed for sharpening, and the consumption of steel is small.

A third modification of the general plan is to fix a properly shaped chisel on the bar of a slotting machine, and by help of it, along with the graduations of the circular table, to groove out the teeth. However, as it is obviously impossible to effect this at one stroke, it is necessary to go frequently round the periphery, advancing the cutter a little at each round. This plan is convenient for internal gear, especially when a thin groove has been turned beneath the inner ends of the teeth, to relieve the head of the cutter.

Whichever of these plans be followed, the primary object is to produce a cut of a given shape, previously determined according to the system of engrainage which we have adopted: and, whatever be that system, the form of the tooth varies from wheel to wheel, so that a separate cutter is needed for each number and for each pitch; and since the form of the cutter must be tested by a templett, or something analogous, a vast number of templetts is required. These templetts are, almost of necessity, dressed by hand, so that, if we wish to attain to great exactitude, we must seek some auxiliary contrivance of another kind.

For small teeth, drawings on paper are out of the question; and even for large teeth, it is prudent to make drawings of colossal magnitude, and to reduce these to the required size by means of an accurate pantagraph, or by help of the miglioscope, which any engineer may construct for himself. Figure 22, Plate X., represents this little instrument, the use of which I cannot too strongly recommend. A B C D represents a long box, closed at one end by a piece of plate-glass A C, against which the templett is placed for examination: it is convenient to have a thin bar and stop laid across this, for the purpose of readily replacing the templett when it has been removed for correction. E F is a slide, carrying in its centre a good achromatic object-glass G, which may be of from twelve to twenty inches focal length. The box ought to be capable of being elongated to twice the focal distance of the object-glass. The enlarged drawing, supported on a small frame, being placed at H, the object-glass is adjusted till the image of the drawing fall on the exterior surface of the plate-glass, and the coincidence of the templett with that image is examined by help of a lens.

It is always safe to test the accuracy of the reduction by means of two scales in the required proportion, the one being laid to the drawing, and the other applied to the glass plate: but the approximate adjustment may be made by help of the following formula:—Let f be the focal distance of the lens for parallel rays, m the magnifying power wanted, then the distances from the centre of the lens to the drawing and to the screen must be

$$(1 + m) f, \text{ and } \left(1 + \frac{1}{m}\right) f$$

respectively. This instrument is only the common portable camera obscura modified to suit our present purpose.

The cutter, having been formed in accordance with the templett, has to be adjusted to its true position on the dividing-engine: its mesial plane must pass along the axis of the index-wheel, and its summit must traverse at the true distance from that axis. The first of these adjustments is often entrusted to the eye, and the second judged of by the fulness of the tooth; but the best expedient for ensuring accuracy, is to run a few cuts into a waster, or into a piece of sheet-brass, and to examine these by help of the miglioscope: on this account the drawings should extend over two or three teeth.

In order to determine the successive angular positions of the blank, it is attached to the arbor of a well divided index-wheel. Watch and clockmakers use a flat index, on which concentric circles, variously divided to suit the different purposes for which they are required, are traced; the divisions being usually given by small holes drilled into the plate. A stiff spring-stop enters into these holes, to retain the index-wheel in its proper position during a cut. This instrument is inconvenient for two reasons: one that, since we have often to use aliquot parts of the numbers given on the index, great attention is required to avoid making a false cut by bringing the wrong hole under the stop: another that, as all the holes are not used alike, the divisions become erroneous by the unequal wearing of the metal. Besides, if any number not marked on the plate be required, there is no ready means for obtaining it.

The snail-index-wheel, now in common use, besides remedying these defects, furnishes facilities of another kind. In this the periphery is notched to fit the convolutions of a screw or snail which turns on an axis perpendicular to that of the index; so that each turn of the snail brings the index one division round. This species of action falls to be considered under the general head of *skewed-gear;* meanwhile we shall suppose that the apparatus is accurately made, so as to produce, from an equable motion of the snail, a motion also equable of the index-wheel. The number of teeth in the index-wheel is usually large, as 100, 120, 200, etc.; let it be denoted by w.

By means of this apparatus we can at once cut a wheel having the same number of teeth with the index, by giving after each stroke of the cutter one turn to the snail-shaft. We can also readily obtain any number which is a divisor of w, by giving to the snail the proper number of complete turns. To produce any other number of divisions as v, we have to proceed somewhat differently. Since one turn of the index is effected by w turns of the snail, it follows, that if we divide the turn of the snail into v equal parts, and, at each cut, count w of those parts, we shall subdivide the blank into v parts, and the errors of that subdivision will be w times less than the errors of the division of the snail. Now the counting of w divisions at each cut is exceedingly laborious, and liable to mistake; it is avoided thus: Having determined on the pitch of our system of change-wheels, and cut a wheel of w teeth to this pitch, we divide a card disc carefully by hand into v equal parts, and fix it on the axis of the snail: then by its help we cut one wheel of v teeth, prepared to fit on the same axis; the error of the subdivision of this wheel will be quite insignificant. We now fix it in place of the divided card, and bring the wheel of w teeth to bear upon its edge: each turn of this last wheel will now move the index through one of the v divisions, and we are prepared at any time to cut a wheel of v teeth. By this plan the trouble of counting is avoided: an arm attached to the wheel w (called the stop-wheel) is allowed to drop into a notch prepared to receive it; and, to avoid changing the bearings of the stop-wheel at each change of the number v, an intermediate wheel of any convenient size attached to a moveable arm is introduced.

Since the stop-wheel, in this arrangement, makes an entire revolution during the transit of a tooth, its angular motion must coincide with the α of our preceding investigations; and if it be subdivided into 40 equal parts, we have a convenient apparatus for projecting the forms of the teeth from the orbits which we have investigated.

The advantages attending the use of this machine are considerable: the resistance to the displacement of the index during a cut is spread over a large surface, and is unaccompanied by any bruising of the parts: the tendency of the wearing is rather to improve than to injure the exactitude: the counting is superseded, and the stop-notch single and unmistakeable. Again, in actual working, we desire to save as much as possible our finishing cutters; we therefore first rough out the wheels by a less valuable tool, which may be a worn-out cutter re-sharpened; now, on replacing the wheels, to subject them to the action of the finishing cutter, the already made opening may not come exactly opposite the tool; but, by changing the position of the stop-notch, the adjustment is made without loss of time. For this reason, as well as for other facilities, the ring carrying the stop-notch ought to be moveable, and furnished with divisions and fixing screws, by means of which it may be secured in any desired position.

Those methods of operation in which the cutter gives at once the true form to the tooth, are applicable to all systems of engrainage, and leave full scope for other considerations in directing the choice of the system. Yet, from the importance of reducing, within the narrowest possible limits, the enormous accumulation of cutters, those systems which admit of the permutation of all wheels of the same pitch, possess paramount advantages.

FROM whatever point of view we contemplate this subject, the interchangeability of the wheels is a primary desideratum, and we now proceed to study the general characters of all those systems of engrainage which admit of it, in order to open up the principles of the second class of entomic methods.

There is no such thing as a pair of wheels suited to each other alone: every pair forms part only of a system. Thus if, from any arbitrarily assumed wheel A, we deduce a conjugate wheel B, these typify two series of wheels so related to each other that each one of the series A may be geared into any one of the series B, and their permutability requires that the two series be identic: now the extreme limit of each series is the straight rack, and therefore the rack belonging to the series A must be an exact counterpart of the rack belonging to the series B. Hence, as we have already seen, the outline of the tooth of the rack must be composed of four equal parts, symmetrically arranged upon the pitch line. Conversely, if the contour of each tooth of a rack be composed of four equal portions symmetrically disposed, it typifies a system of interchangeable wheels.

Let figure 23, Plate XVI., R W be the pitch line of a rack, R V being the pitch or distance from tooth to tooth. Divide R V into four equal parts at S, T, U; raise the perpendicular S X; assume any outline R X for the fourth part of the contour of a tooth, and repeat that curve in the positions T X, T Y, V Y, V Z, etc.; then R X T Y V Z etc. is the outline of a rack which will generate a system of permutable wheels.

Having made a cutter with X T Y V Z for its outline, and mounted it on a carriage provided with a traverse motion and leading screw; let us connect this leading screw with the stop-notch ring, in such a way that one turn of the ring may give a traverse motion equal to R V the distance between two teeth. Then, the cutter being arranged so that its pitch line may pass at the proper distance from the axis of the index wheel, bring up the cutter until in descending it slightly graze the edge of the prepared blank, and make, in the ordinary way, a series of cuts all round. By turning the notch-ring a little forward, bring the cutter deeper on the blank, make another series of cuts: again advance the notch-ring, and continue this process until the cutter pass entirely clear of the metal, then shall we have a wheel whose teeth are truly formed to work with our assumed rack; and all wheels formed in this way will gear truly into each other.

By following this plan, the machine maker can accommodate his own taste or judgment in determining primarily the shape of the rack-tooth; and, in order to preserve uniformity in his results, he has only to secure the means of reproducing cutters exactly of the same form with the original one. All wheels of one pitch are thus cut by one tool, and, as the machine can easily be rendered self-feeding, the operation, though apparently tedious, need not be very expensive. This process is inapplicable to internal gear. When the assumed form of the cutter is the curve of sines, the wheels belong to one or other of the systems of which we have already treated.

From the arbitrarily assumed form of the rack we can deduce the motion of the tracing points and the number of contacts.

Let P be any point in the outline of the rack, and let it be required to find at what time the contact will take place there. Draw Pq normal to the curve at P, and cutting the pitch line at q, then, when q is at the pitch point Q, the contact will take place at P; in other words, Rq gives the value of α corresponding to the contact at P. Since the normal Pq can cut the pitch line only once, it follows that each point in the contour of the rack can only come once into action; but it does not thence follow that the point q can only occur once.

To resolve the converse question; that is, to discover at what points the contact takes place when q is at Q, we must draw from q a line qP normal to the curve: now it may be possible to draw several normals as qP, qP′, qP″, etc. and, in that case, there must be so many points simultaneously in contact.

If we assume some fixed point Q to represent the pitch point, and draw from it Qp, Qp', Qp'', etc., equal and parallel to qP, qP′, qP″, etc.; and then having shifted q, repeat the operation until q shall have passed along the whole line R V, the extremities p, p', etc. will mark out the entire orbit of the points of contact. When the successive positions of q are taken at equal intervals along R V, we are enabled conveniently, by the method already explained, to delineate the form of a wheel of any proposed number of teeth.

Should the curve R X T Y V Z, etc. be so flat that the centre of curvature for any point P lies on the opposite side of the pitch line, then, if P begin from R and gradually pass along the contour of the rack, the point q, also beginning at R, will describe continuously the pitch line R W without ever retrograding, and, in this case, there can never be more than one contact at a time.

But if the tooth be deepened until the centre of curvature for the vertex X lie on the same side of the pitch line, the point q will reach S before P has reached X; q will then pass beyond S, become stationary, and then retrograde to reach S again, just when P comes to X: still continuing its retrograde motion, q will pass above S, become stationary, and then advance to reach T at the same instant with P. A similar retrogression will occur about U. In this manner a part of the pitch line lying equally on each side of S is described three times, viz. twice forwards and once backwards, by the point q; and from every point taken within this limit, there may be drawn three normals to the curve corresponding to one plerotic and two pragmatic contacts.

If the tooth be gradually deepened, the range of retrogression will expand and come at length to occupy the whole distance R T, while the retrogression about U will occupy the whole distance T V: at this limit, the point q will pass three times along the whole pitch line, and there will always be three and just three points of simultaneous contact.

Continuing still to deepen the tooth, we bring the ranges of retrogression to overlap each other about T, and the region of the overlapping will be passed over five times by the point q, so that there will be sometimes three and sometimes five contacts; and if the tooth be still deepened until the overlappings reach the points S and U, there to be met by those belonging to other waves of the curve, the pitch line will be traversed five times by q, and there will be constantly five simultaneous contacts.

In this way the depth of tooth necessary to give any prescribed odd number of contacts may be investigated. Supposing P to be confined to the quadrant R X, if R q, the sum of the absciss R H and the subnormal H q, have no maximum value, or if its maximum value be just R S, there will be only one point of contact; if the maximum value be R T, there will be three contacts; if it be R U, there will be five contacts, and so on; the number of contacts being one less than the number of eighth parts of R V contained in the maximum value of R q; and the number of teeth fully engaged being indicated by the number of times which the same maximum value contains the half pitch R T.

These conclusions only hold good when the outline of the rack is free from undulations and of transitions *per saltum*. They are analogous to those which we have already obtained, but are more comprehensive, in as much as they hold good for every practicable form of tooth. On being applied to the curve of sines, they give formulæ identic with those which we have already used in computing Table I.

If we draw normals to the outline of the teeth from the pitch point, the extremities of these normals are the simultaneous points of contact.* So long as we confine our attention to the contour of the straight rack, there is nothing particularly remarkable in this theorem: but when we proceed to apply it to the outline of a wheel, it becomes somewhat paradoxical. It is possible to draw normals to the outlines of the teeth, even on the opposite side of the wheel, and it seems obvious that the extremities of these cannot be points in the orbit; yet the demonstration of the theorem is quite universal, and involves no limitation either as to the size of the wheel or as to that side of it which must be taken. Some ulterior principle must lurk in this apparent incongruity.

Instead of having assumed the rack as the prototype of our system, we might have assumed any one wheel. Thus, to give the subject a practical aspect, having obtained a pinion-wire, we may require to cut out the teeth of a wheel which may be led by it.

The mechanical resolution of this problem is analogous to the second method of cutting wheels; though of little practical utility, the consideration of it may serve to extend and generalize our ideas. Let the assumed form of the wheel be extended into a cylindroid as in the pinion wire; let it be notched so as to form a series of cutting edges as on the bar cutter, and let it be fixed in such a manner as to slide parallelly to the axis of the index wheel, while it may be turned upon its own axis by means of an index plate. Having brought the pinion wire down upon the prepared blank so as to make a cut, repeat this in the usual way all round. Turn now the pinion wire on its own axis by a fraction of a tooth, and at the same time turn the stop ring by a similar fraction of a revolution; repeat the cut all round in the usual way, and continue this until the stop ring have made a complete revolution, then is the blank toothed in such a way as to gear truly with the pinion.

Suppose figure 24, Plate XVII., to represent the arbitrarily assumed form of a wheel, and let it be required to discover the law of the motion of the tracing points in the system of which it is the prototype.

Having taken any point P in the outline of the wheel, draw through P a normal to the curve: the intersection of this normal with the pitch line will give the corresponding position of the pitch point. When P is outside of the pitch circle there may happen to be no intersection, in which case that part of the outline can never come into contact: the limit of possibility is when the normal just touches the pitch circle, and then the contact occurs in the line D Q E. In all other cases the normal to the curve at P must cut the pitch circle twice, as at q and q', and our problem, in reality represented by a quadratic equation, has two solutions.

In order to delineate the orbit of the tracing point, we must turn the wheel until q come to Q, and then note the position occupied by P; this, repeated for points taken all along the outline, will give the required orbit. Or we may bring the other intersection q' up to the pitch point. In this way we shall trace out two distinct curves, each involving its own law for the motion of the tracing points, and each typifying a system of wheel-work of which the assumed wheel is a member.

If for one of these orbits we put P for the distance Q P, O for the angle of obliquity, and α for the length of the pitch line; and for the other orbit P'; O', α'; r being the pitch-radius of the assumed wheel, then shall we have

$$\left.\begin{aligned} O' &= -\,O \\ P' &= +\,P - 2\,r\sin\,O \\ \alpha' &= +\,\alpha - 2\,r\,O \end{aligned}\right\} \quad . \qquad (32).$$

Here if O, P, α satisfy the condition of contact which is expressed by the equation $\partial P = \cos\ O \,.\, \partial\alpha$, the quantities O', P', α' will satisfy the analogous condition, as is evident on taking their differentials.

Each orbit of the tracing point represented by (O, P, α) is thus accompanied by an infinity of co-related orbits (O', P', α', r), each of which generates a system of wheels, and that wheel whose pitch-radius is r is common to two of the systems.

To exhibit this very singular property more clearly, let the knot at Q, figure 25, Plate XVII., purposely made unsymmetric, represent the orbit of the tracing point, and let Q C = r be the radius of the pitch line. Then the motion of the tracing point in the knot Q, combined with the proper angular motion round C, will give the form of the tooth belonging to the proposed system. Having made an exact copy of the knot Q, let it be obversed, in order to satisfy the equation $O' = -O$, and let the point q be carried round the circumference of the pitch-line, the line $b\,q\,a$ preserving its parallelism throughout. Let also a straight line $Q\,p\,q\,p'$ constantly connect the two points Q and q, then the intersections p and p' of this line with the moveable knot will trace out a new orbit for the point of contact.

The law of the motion of the tracing point in this new orbit having been determined either by help of equations (32) or by the condition of contact, a new system of wheels will result: the teeth, however, are found on the wrong side of the centre, and therefore, although the contact take place geometrically, the wheels are mechanically impossible, or the system is apragmatic. Nevertheless, the wheel of r teeth belonging to the new, is identic with the wheel of r teeth belonging to the original system.

The examination of the above paradox has thus conducted us to a very singular generalization: it also opens up other lines of inquiry.

The quantities O, P, α being so related that their variations satisfy the equation $\partial P = \cos O \,.\, \partial \alpha$, we have seen that from these there may be obtained three other quantities O', P', α', satisfying the same condition. There is, indeed, an infinity of ways in which these variables may be modified, and the general inquiry into the law of modification is interminable. Those classes of changes in which O, or rather cos O, remains unchanged, are worthy of being studied, as offering some practical results.

The truth of the equation $\partial P = \cos O \,.\, \partial \alpha$, is not affected by any change on the values of P and α which does not alter their differentials, and, therefore, we are at liberty to increase or diminish each of them by any constant quantity. The substitution of $\alpha + c$ for α would merely change the point from which we begin our reckoning, without in the least degree altering the form of the tooth: but the substitution of $P \pm C$ for P produces a new set of phases. Thus if, having joined Q P, figure 4, Plate II., we lay off from P a constant distance $P\,P'$, either inwardly or outwardly, and, while P moves along the orbit, if we observe the curve traced out by the new point P', this curve will be another orbit, such that the simultaneous motions of P and P' will produce odontoids separated from each other by a constant distance $P\,P'$, measured on their common normal. Imagine one wheel A produced by the motion of P, and another wheel B produced by the motion of P': the peripheries of these wheels, instead of touching each other, will be separated by the distance $P\,P'$, at what otherwise would have been their points of contact.

Let us suppose α to be augmented by some quantity φ, a function of O and P; in this case we have $\alpha' = \alpha + \varphi$, whence, since $\partial P' = \cos O \,.\, \partial \alpha$,

$$\partial P' = \partial P + \cos O \,.\, {}_{1O}\varphi \,.\, \partial O + \cos O \,.\, {}_{1P}\varphi \,.\, \partial P;$$

or, taking the integral,

$$P' = P + \int \cos O \,.\, {}_{1O}\phi \,.\, \partial O + \int \cos O \,.\, {}_{1P}\phi \,.\, \partial P. \quad (33).$$

The second of these integrals can only be obtained by help of the actual orbit, or of the relation between O and P, so that it can hardly enter into any general system of modification: hence, for our present purpose, it is enough to consider the case in which the derivative ${}_{1P}\phi$ is zero, that is, in which ϕ is independent of P. Our equation then becomes

$$P' = P + \int \cos O \,.\, \partial \phi,$$

that is, we are at liberty to augment α by any monome function of O, such as ϕ, provided we, at the same time, augment P by the integral of $\cos O \,.\, \partial \phi$. The general character of this modification may be very clearly exhibited thus. Let O, P, α, belong to any orbit, O, (P), (α), to any other orbit, then a third orbit O, P$'$, α', may be obtained by making

$$P' = P + (P)\,;\ \alpha' = \alpha + (\alpha). \quad (34).$$

The co-related orbit which we have had occasion to notice, is one of this class of modifications.

To return, from this apparent digression, to the methods of cutting out the teeth of wheels, we observe that, in both of the above classes of methods, the accuracy of the workmanship depends essentially upon forming a cutter truly to a prescribed shape. Now, however truly a piece of soft steel may be shaped, it can hardly be hardened without warping; and, as it is gradually blunted by use, it requires to be sharpened while hard; hence, if we look for extreme precision, our success will depend entirely on our means of accurately shaping a piece of hard steel.

The straight and the circular are the only two forms which we can readily and certainly give to a hard steel edge. The first we can obtain by means of the flat lapp; and the second by causing the cutter to revolve while its edge is brought up to the grinding tool: or when a segment only is wanted, by grinding it on a previously dressed spherical surface. But the straight edge will not answer our purpose, as it cannot be applied to the concave parts of the tooth, and therefore, when we desire to obtain the utmost possible precision, we must have recourse to the circular cutter.

A very convenient arrangement of this tool is represented in figure 26, Plate II., where a is the cutter fitted truly upon a centre-pin, so that when one part of the edge has been blunted another part may be brought round without altering the cut; b is a pinching-nut for securing the cutter in its place, and c the stock or tool-holder to be fixed in the slide-rest. By help of this cutter we can obtain, with almost any required degree of precision, the true outline of the teeth.

Having fixed it in the slide-rest, or other co-ordinate instrument fitted with minute graduations, we can give to its centre any computed position, and, having made a cut, we can shift it into a contiguous position, and so continue until the whole outline be passed over: this process is practicable for any curve, provided the radius of the cutter be less than the least radius of curvature of the concave part of the desired outline. By this operation we obtain a true general outline, but not a smooth curve. Instead of an even surface we have a succession of cylindric grooves, the bottoms of which touch the outline which we desire to form; yet, by bringing the cuts closer together, we have it in our power to diminish the asperities until they occasion no practical inconvenience.

In adapting this method to the cutting out of wheels, we may follow either of two distinct processes, analogous to those which we have already described.

In the first of these we cause the edge of the circular cutter to pass along the outline of the tooth, by computing the co-ordinates, not of the odontoid itself, but of another parallel to it, obtained by laying the radius k of the cutter back upon the line P Q. Then, by help of the graduations of the slide-rest, we bring the centre of the tool into one of the computed positions, and make a series of cuts all round the blank, or rather the roughed-out wheel. We shift the cutter into the next computed position, make another round of cuts, and so continue until the whole contour be shaped out. For the previously investigated systems of odontoids, the radius of curvature at the bottom of the rack-tooth is less than that at the bottom of the tooth of any external wheel, and hence that radius may be taken as the limit to the size of the cutter: its value, $k = \frac{1}{Y}$, adds another to the geometrical progression, B M, 2 X, Y, R, k. The numerical values of k are given in Table I. under the proper head.

This tedious operation is rendered still more irksome by the necessity of making a separate computation for each size of cutter; for when the tool is blunted, another exactly of the same size must be prepared, or else a new computation must be made. In this form the process is almost impracticable; but, by help of a little contrivance, it may be rendered tolerably convenient. It deserves the more attention, that it is applicable to the formation of curves in general.

Having placed an accurate drawing of the desired outline securely in a position parallel to the work, and attached a microscope to the cutter carriage, we bring, by help of the leading screws, the cross-wires of the microscope along the drawing; then the centre of the cutter describes the required curve. This, however, is not what we want: the radius of the cutter has yet to be allowed for. Instead of fixing the micrometer permanently to the tool-carriage, we attach it by means of a circular plate to which we give such an eccentricity that the crossing of the wires may describe a circle exactly equal to the edge of the cutter: this is at once ascertained by placing the cutter itself under the microscope, and turning the eccentric wheel round, when the crossing of the wires ought to pass along the edge of the cutter. These wires should be placed so that one may be radial, the other tangential. If, now, we replace the cutter and carry the crossing of the wires carefully along the drawing, observing to keep the tangent wire always in the direction of the curve, the circumference of the cutter will trace out an exact copy of the same curve. For a drawing of the actual size we may substitute the image of a large drawing reduced by help of a lens, and may thus obtain an extreme degree of precision.

It is well known to those in the habit of using the slide-rest, that, in order to produce a true surface, the finishing cut ought to be as nearly as possible of the same depth throughout; hence, if we desire to be exceedingly precise, we may place in the field of the micrometer a third wire, *c*, figure 27, Plate VI., parallel to the proper tangential wire *b*, and may carry the intersection of *c* with *a* along the curve, so as to prepare for the finishing cut, during which the intersection of *b* with the radial wire *a* must be used.

This method requires an accurately made drawing of the tooth; it is considerably more tedious than the second method, because the machine cannot be made self-feeding; but it is susceptible of an extreme degree of precision. The chief inconvenience is that a drawing, and, of course, a previous calculation, must be made for each specific size of wheel.

Instead of carrying the intersection of the wires along the outline of the tooth, while the stop-notch remains fixed, we may carry the same intersection along the path of the tracing point, provided the radial wire *a* be always directed towards the pitch-point Q, and provided the position of the stop-notch be shifted according to the successive values of α. To render this fourth method convenient in practice, we may compute the successive values of ω, and graduate the circular motion which carries the micrometer. Or otherwise we may draw, from each of the successive positions of the tracing point, a short line directed towards Q. The latter method, particularly when the figure of the orbit is reduced optically, will be found the more convenient.

By adopting this fourth method, we avoid the necessity of drawing out each wheel, and can at once proceed, by help of a table of the co-ordinates of the tracing point, to cut teeth according to any system which we may have adopted. Though requiring more skill in the management, it is not greatly more tedious than the third method; and therefore, when the required degree of precision exceeds the capabilities of the first method, the fourth should be had recourse to.

The outer series of plerotic contacts is first to be passed over, then the right and left hand pragmatic contacts, and lastly the inner series of plerotic ones: but, for the finishing cut, it is as well to begin at the point of the tooth, and thence to pass continuously over the whole outline.

Thus it appears that the true shape of a wheel can only be obtained with great labour, and that there is no empiric rule which can answer in all cases.

Having now examined the general characters of the four practicable entomic methods, we are prepared to search, among the infinity of odontoidal systems, for those which give facility to the application of one or other of these methods.

In the first and third of these entomic methods, the form of the tooth is given directly, in the one by a cutter of the proper shape, in the other by causing the cutter to describe the required curve. In the second method, a rectilineal motion is combined with an angular one, and the character of the wheels depends on the form of the cutter, which represents the tooth of the straight rack: while in the fourth method the tooth is formed by combining the motion of the tracing point in its orbit with the angular motion of the wheel. Hence, while searching for forms which may be easily cut by the first, or the third method, we must look for simplicity in the shape of the tooth: while using the second process, we give our attention to the form of the rack; and, in employing the third method we seek for simplicity in the orbit of the tracing point. The inquiry before us thus separates into three branches.

Though we were to assume some simple or easily procured form for the tooth of one wheel, which should be the prototype of our system, the forms of the other wheels of the series would be complex, and thus the first branch of our inquiry may be dismissed as likely to be barren of any useful result.

The second branch of our investigation relates to the form of the rack-tooth: now the outline X T Y V Z, figure 22, shews the general form of the cutter, the only requisite being that the four quadrants X T, Y T, Y V, Z V, be equal to each other, and symmetrically placed. If, then, the part T Y be convex, the part T X must be concave, and it will be difficult to form its outline otherwise than by hand: the part of the ogee about T, however, is nearly straight; let us inquire what would be the effect of making the side of the cutter straight throughout.

Suppose R X T Y V Z, etc., to be the form of a rack obtained by drawing through R, T, V, etc., straight lines equally inclined alternately to the right and to the left. To find the law of the motion of the points of contact in the system to which this rack belongs, assume in the pitch line any point q, and thence draw the normals q P, q P′ to the sides of the teeth; then, when q is brought to Q, P and P′ will indicate points in the required orbit. Draw then Q G, Q F, figure 28, Plate XVIII., parallel to q P, q P′, and measure off Q p, Q p' equal to these normals; p and p' are simultaneous points of contact. Now, wherever q may be taken, the angles R q P, R q P′ are constant, therefore the tracing points describe two straight lines equally inclined to Q R, the one to the right, the other to the left. Also, if we put i for the constant inclination, and observe that, according to the notation which we have adopted, R q = α, R T = π, we have

$$Qp = \alpha \cos i; \; Qp' = (\alpha - \pi) \cos i \quad . \quad (36)$$

equations which indicate an indefinite scope to the motion of the tracing points.

If, from C, the centre of any wheel whose pitch radius C Q is r, we draw C L perpendicular to Q G, the linear velocity at the distance C L is just equal to the velocity of the tracing point in the line Q G, and hence the outline of the tooth must be the involute of the circle whose radius is C L $= r \cos i$: now this curve extends indefinitely both ways, and there is no geometric limit to the motion of p; but there is a very obvious limit to the mechanical contact.

Having raised at X the line X q' perpendicular to R X, it is evident that the actual contact ceases when q' comes to Q; and therefore, if we measure off Q G, Q H, Q I, Q F, each equal to q'X, we have the limits to the paths of the tracing points.

Since the contacts must recur in similar positions after the passage of each tooth, there must be an infinity of tracing points distributed along each of these lines, and moving forwards simultaneously in such a way, that one of each series alternately reaches the point Q: the positions of these points are thus represented:

$$Qp = \alpha \cos i; \; (\alpha - 2\pi) \cos i; \; (\alpha - 4\pi) \cos i, \text{ etc.}$$

$$Qp' = \qquad (\alpha - \pi) \cos i; \; (\alpha - 3\pi) \cos i, \text{ etc.};$$

so that the distance from one tracing point to another of the same series is 2π cos. i.

Now R S = $\frac{1}{2}\pi$, wherefore G H = π . cos. i . cse. i^2, and the distance between two consecutive contacts is contained in the whole range H Q G + F Q I, cse. i^2 times. When cse. i^2 is an integer number, there will always be that number of contacts on both the lines H G and F I: and if this be an even number, there will be as many contacts backwards as forwards, so that, since these contacts are all pragmatic, a constant number of teeth will be fully engaged. Hence, if we wish to adopt the system in which t teeth shall always be at work, we must make

$$\text{cse } i = \sqrt{2}\, t; \quad . \quad (37)$$

wherefore, to obtain the complete action of two teeth, we must use an inclination of 30°.

These contacts are all pragmatic, and, at first sight, the system would seem to include no others. However, that very circumstance which limits the range of the working contacts calls into existence a set of supplementary ones. Thus, while the point of contact is passing along the side of the rack-tooth, the apex X is describing upon the blank disc a curve line which marks out the bottom of the tooth, and indicates the region of plerotic contact. And although the top and bottom of the tooth of the rack be contracted to a point, those of the wheels are not so. Hence, in addition to the two series of points which progress along the lines H G and F I, we have two other series describing F G and H I, with the velocity of the pitch lines, and separated from each other by intervals of 2π: one of each series comes alternately to the line of centres.

The value of G F is clearly $\pi . \cot i^2$, and therefore the number of intervals repeated along F G must be $\frac{1}{2} \cot i^2$; but the number of intervals repeated along F I is $\frac{1}{2}$ cse i^2, wherefore there is exactly half an interval more between F and I than between F and G: thus the two contacts which started simultaneously from F, do not reach G and I at the same time, but the plerotic contact disappears one-half of an interval earlier, that is, exactly along with some one of the pragmatic contacts which had traversed the line H G. The reader would do well to compare this progress of the tracing points with their motions in the hour-glass curve. The number of intervals in the sum of F G and H I is exactly one less than the number in the sum of F I and H G; so that, while the plerotic contacts are always less numerous by one than the pragmatic, the entire number of contacts must always be odd.

Such being the general phases of the orbit of the tracing point, it remains for us to examine the positions of the metacentres, or the limits at which the outline of the tooth is convoluted.

Since the velocity of the tracing point along F G is exactly equal to that of the pitch-line, it must be greater than the linear velocity of any disc whose centre C is on the same side of the pitch-point; wherefore its trace must be forwards on the disc, and there can be no convolution at the bottom of the tooth.

Whenever the perpendicular C L falls within Q G, there is a reflexure on the side of the tooth, since the tracing point, in describing H L, is approaching to C, and, having reached its minimum distance at L, must afterwards recede: L indeed shews the cusp of the involute, and it is obvious that the second branch of the curve can have no mechanical existence. Draw, then, G N parallel to R X, and we have at once the position of the first metacentre, or that point at which the reflexure disappears from the sides of the tooth. Since Q N = Q G cse i, we have

$$\mathrm{Q\,N} = \frac{\pi}{2}\,\mathrm{T}\,\sqrt{(\mathrm{T}-1)}, \qquad (38)$$

where T is the number of pragmatic contacts.

The velocity of the disc near the point of the tooth being, in all cases, greater than that of the pitch-lines, must be greater than the velocity with which H I is described; therefore the trace made there must be backwards upon the disc. Hence, however great may be the radius of a wheel constructed according to this plan, there is always a replication at the point of the tooth; in other words, the metacentre M is removed to an infinite distance. Also, since the replication always includes part of the involute, the pragmatic contacts are deficient, or the metacentre O is also at an infinite distance. On this account any general investigation into the number of actual contacts fails, and we are forced to content ourselves with the computation for each particular pair of wheels.

The form of the orbit for the system of seven contacts is given in figure 29, Plate XIX., and the shape of a pinion of twelve leaves thence deduced is shewn in figure 30, Plate XX., with the replication at the point of the tooth.

THE inconvenience of this replication is avoided by truncating the teeth of the rack, in order to diminish the region of plerotic contact at the bottom and augment that at the point of the wheel tooth. In this way the metacentre O may be brought within a finite distance.

Let, in figure 31, Plate XXI., $\mathrm{X}x = \mathrm{Y}y = \mathrm{Z}z = \varphi$ be the extent of the truncation; then, having drawn X S perpendicular to R T, and X Q perpendicular to R X, we have

$$\mathrm{S\,X} = \frac{\pi-\varphi}{2}\cot i;\ \mathrm{G\,H} = (\pi-\varphi)\cos i\,.\,\mathrm{cse}\ i^2;$$

$$\mathrm{F\,G} = (\pi-\varphi)\cot i^2.$$

Now the distance between two consecutive contacts on the line H G is $2\,\pi \cos i$, wherefore the number of intervals contained in that line is

$$\frac{\pi-\varphi}{2\,\pi}\,\mathrm{cse}\ i^2;$$

similarly, the number of intervals on F G is

$$\frac{\pi-\varphi}{2\,\pi}\cot i^2;$$

thus the number of pragmatic contacts no longer exceeds the number of plerotic ones by unit, but by $\frac{\pi - \varphi}{\pi}$.

If we desire to have always a given number T of pragmatic contacts, we must determine i from the equation

$$\text{cse}\ i = \sqrt{\left\{\frac{\pi T}{\pi - \varphi}\right\}}; \quad . \qquad (39)$$

but then the number of plerotic contacts will not be T — 1, but sometimes T, and sometimes T — 1, these being the two integer numbers between which $T - \frac{\pi - \varphi}{\pi}$ is included.

Equation (39) may be put under the form

$$1 = T \,.\, \sin i^2 + \frac{\varphi}{\pi},$$

in which i and φ appear as functions of each other; thus the amount of truncation being determined on, we can ascertain the obliquity necessary to give a certain number T of pragmatic contacts. It is clear that, in all practical cases, of the two variables φ and i, the one increases while the other decreases; also, since the metacentric distance Q N is given by the equation

$$QN = \frac{\pi}{2} T \cot i,$$

Q N increases with φ, while, as a little reflection will convince us, the metacentric distance Q O decreases.

Now, if we endeavour to obtain wheels of small numbers, by augmenting φ for the purpose of reducing the value of Q O, we at the same time augment the value of Q N; therefore the optimum arrangement must be when φ and i are taken so as to suit the equation

$$QN = QO, \quad . \qquad (40)$$

in which case all wheels having their pitch-radii greater than Q N, will have their pragmatic actions complete.

Let T Y y V, in figure 32, Plate XXII., represent the interval between two teeth of the rack, in the position corresponding to $\alpha = 270°$ (position 30 of our figures), Q occupying the middle of T V. From C, the centre of any wheel, draw C L parallel to Y V, and through Q draw H P Q L perpendicular to the same line, H being in the continuation of y Y; describe a circle with the radius C L, and, by bending and unbending the tangent P L, trace the involute A P B, continuing it until the tangent B E be equal in length to H L: then is B P that part of the outline of the tooth which has been in contact with Y P. Should the curve A P B be found to cross the line of centres C Q, the symmetric curve traced from T on the other side would also cross it, and thus there would be a replication: but if A P B be all on one side of C Q, there can be no replication. Hence, if B fall just upon the line of centres, the centre C must coincide with O, which indicates the limit of complete pragmatic action at the points of the teeth.

Describe through Q the involute Q D, so that the arc L D may be equal to the tangent L Q, join C E, C B; and put r for the radius C Q. Then, since $(\pi - \varphi)$ cse $i^2 = \pi \mathrm{T}$, we readily obtain

$$\mathrm{LH} = \mathrm{EB} = \mathrm{EA} = r \,.\, \sin\, i + \tfrac{1}{2}\pi \mathrm{T} \,.\, \cos\, i; \text{ whence}$$

$$\mathrm{ACB} = \tan\, i + \frac{\pi \mathrm{T}}{2r} - \tan^{-1}\left\{\tan\, i + \frac{\pi \mathrm{T}}{2r}\right\}:$$

also $\mathrm{QCD} = \tan\, i - i$, and D C A, is evidently the quarter of the interval between two teeth; wherefore $\mathrm{ACQ} = \tan\, i - i + \frac{\pi}{2r}$.

The position of the metacentre O will be attained when the angle A C Q is equal to A C B; that is, when

$$i + \frac{\pi}{2r}(\mathrm{T}-1) - \tan^{-1}\left\{\tan\, i + \frac{\pi \mathrm{T}}{2r}\right\} = 0; \quad . \quad (41).$$

In order to combine this with condition (40), we must insert for $2r$ the value $\pi \mathrm{T} \cot\, i$, which gives

$$i + \frac{\mathrm{T}-1}{\mathrm{T}} \tan i = \tan^{-1}\{2 \tan i\}; \quad . \quad (42)$$

an equation which involves only T and i. By help of it we can compute directly the value of T for any given value of i; but the converse computation can only be performed by trials.

This relation between i and T may be readily exhibited thus: Having described a circle with the radius M G, figure 33, Plate XXX., make the angle G M Q equal to i, apply the tangent G Q, and measure off Q H equal to G Q, join M H, cutting the circumference in D, then we ought to have

$$\mathrm{QH} : \mathrm{ED} :: \mathrm{T} : \mathrm{T} - 1.$$

The values of i for each odd number $2\,\mathrm{T} - 1$ of contacts having been computed by successive approximations, those of Q M, Q G, and φ, entered along with them in table 8, were obtained.

Figure 33 having been constructed from these dimensions, actually for the case $\mathrm{T} = 2$, the point of the tooth belonging to the pinion whose centre is M, falls at B in the continuation of the line M Q. Now M B is equal to M H, and therefore B must fall beyond the line of truncation of the rack; so that, to make room for the motion of the point of the pinion tooth, the bottom of the rack must be scooped out to B. The arc H B I, described from the centre M, shews the motion of the point of the pinion tooth, which must trace upon the plane of the rack a curve Y B *y*, part of a curtate cycloid, formed by rolling a circle, whose radius is M Q, along the pitch line. It is also necessary, for the permutability of the wheels, that the points of the rack-teeth be truncated, not by a straight line, but by a similar cycloidal arc. In this way the plerotic contacts, instead of occurring upon the straight lines H I and F G, occur along the circular arcs H B I and F A G.

Since H Q : E D :: T : T — 1, and since the linear velocity of the line D E G is just equal to that of the tracing point along H Q G, the times occupied in describing the lines H Q and D E must be as T : T — 1; but, while the point E describes the arc D E, the point B just describes H B; wherefore the time in which one tracing point describes H Q is to the time in which the other describes H B, also as T is to T — 1; that is, since there are T intervals upon the two lines H G and F I, there must be exactly T — 1 intervals upon the arcs H I and F G: thus the graduation is again quite analogous to that of the hour-glass curve. It is worth notice that the three metacentres N, M, O are collected in one point.

Figure 34 exhibits the form of the compound orbit for five contacts graduated to shew their simultaneous motions; and figure 35 gives the form of the straight rack and of the smallest wheel of this system.

In like manner figure 36 is the orbit for seven contacts, while figure 37 exhibits the two extremes of the system, viz. the rack and the wheel of 17 teeth.

It thus appears that the outline of the cutter for the second entomic method cannot be altogether composed of straight lines; but the curve occurs only in the region of plerotic contact, where an error is of little moment, so that teeth of the truncated involute form can be obtained with considerable precision, and without excessive labour. In this system the sides of the teeth are involutes, the tops and bottoms are curtate epicycloids.

It remains for us to enquire what results will flow from the application of the fourth entomic method; the orbits of the tracing points being taken as simple as possible.

Of all motions, uniform rectilineal is the simplest and uniform circular motion the most easily obtained. Let us examine whether these offer any facilities in the formation of toothed wheels.

We have shewn that the tracing point can only describe a straight path uniformly, when that path is perpendicular to the line of centres, or when it passes obliquely through the pitch point: there are thus two cases of uniform rectilineal motion.

Having assumed some point F in the line of centres, draw the perpendicular G F H, figure 38, Plate XXVII.; then, if the tracing point P describe the line G F H with a velocity equal to that of the pitch lines, its traces made upon all the revolving discs must touch each other. These traces present various aspects, according to the relative positions of the three points, Q, C, and F. When C coincides with F, the curve is the well-known *equable spiral;* when F is at Q, the trace becomes the involute of the pitch circle; and, when F is at the opposite extremity of the diameter of the pitch circle, we obtain a curve which may be called the *companion to the involute,* since its polar equation is formed by changing a sign in the polar equation of the involute: it is remarkable that each of these three curves is invariable in form. In the general position of F, the curve bears to the involute the same kind of affinity that the prolate or curtate epicycloid bears to the epicycloid itself. The gradual variation of form, as the position of F is changed, is exhibited in figure 39, a, Plate XXVIII., in which the line starting from C is the equable spiral, that from Q the involute, and that from G the companion to the involute. Figure 39, b, Plate XXIX., shews the form which the curve assumes when F and C are on opposite sides of the pitch point. The young engineer may exercise himself in computing where F must be placed in order that the curve may be straight at its vertex.

The very genesis of these curves shews that the tooth of the straight rack is a mere point; and therefore none of them can be admitted in practice, since the whole intensity of the action would be concentrated on one spot, and the wear there would be excessive. If, however, we join Q P, figure 38, and cut off from it a constant distance $P P' = P P''$, the points P′ and P″ will describe the curve well-known as the conchoid of Nicomedes, and their traces upon the revolving discs will all touch each other, while the form of the tooth of the straight rack will be the circle described with the radius P P′. So long as the line described by the centre of this pin-tooth does not pass through Q, the points P′ and P″ travel on its circumference; but when F is at Q, the point of contact is permanent in respect of the rack, and the curve is again the involute. As this is the only case which can give results symmetric in regard to right and left hand wheels, it deserves a closer examination.

Let A and B, figure 40, Plate XXX., be the centres of two wheels of which the pitch-point is Q, and suppose that the tracing point has described the distance Q P; then, having measured off, along the circumferences of the two pitch circles, the arcs Q α and Q β, each equal in length to Q P, α and β are the beginnings of the two involutes α P a, β P b, of which the portions α P and β P have been traced by P during its progress from Q. Now each of these curves has Q for its centre of osculation, and Q P for its radius of curvature; hence, if one tooth be convex, the other must be concave, and that during the entire motion; so that, if the motion were reversed, the concave tooth would hook in the convex one and come to a dead stop at Q: there, indeed, the other branches of the curves represented by α a′ and β b′ would come, geometrically speaking, into contact, but, mechanically, they would efface the former branches. Not only so: since the curvature increases from P to α, and decreases from P towards b, the arc P b must lie beyond P α, and, for a similar reason, P a must lie beyond P β, so that the two curves, osculating each other at P, necessarily cross: the mechanical contact of two such surfaces is impossible. We thus see that the motion of the tracing point along D Q E leads to impracticable results.

Having failed in obtaining any useful result from the uniform motion of the tracing point along a straight line perpendicular to A B, we proceed to the only other case of uniform rectilineal motion, that along a line drawn obliquely through Q. This case has already been considered in reference to permutable wheels: in what follows, therefore, we shall pay no attention to the condition of interchangeability.

Let H Q G, figure 41, be a straight line passing obliquely through Q; from A and B, the centres of any two wheels, let fall upon it the perpendiculars A G and B H, and describe circles with these perpendiculars as radii: then, because P describes H G with the linear velocity of either of these circumferences, its traces upon the two revolving discs must be involutes of these two circles.

When the tracing point is situate between H and G, the curves are convex to each other, and the mechanical contact is possible; but when P is beyond G H on either side, it has passed to the other branch of one of the curves, and the mechanical contact is imaginary. Hence H G is the only useful part of the path of the tracing point.

Having measured off G α equal to G P; H β equal to H P, and described the involutes α P K, β P L, these shew the phase of contact corresponding to the position P of the tracing point. From A with the radius A H, and from B with the radius B G, describe two circles intercepting from the involutes the arcs α M and β N, then these portions, which are formed by uncoiling arcs equal in length to G H, are the utmost useable parts of the two curves.

The action of the tooth which commenced when P β M were simultaneously at H, must cease when N, α, P coalesce at G, and then another tooth must either have begun its action or be ready to take up the contact. Hence, by measuring off P P′, P′ P″, etc. $\alpha\alpha'$, $\alpha'\alpha''$, etc. $\beta\beta'$, $\beta'\beta''$, etc., each equal to G H, we obtain the most remote positions of the consecutive tracing points and teeth, in order that at least one tooth may always be in action. If we desire to have two teeth constantly in action, the distance between the successive tracing points must not exceed the half of G H; and, in general, if we wish to have t teeth in action, the distance of the tracing points must not exceed the t^{th} part of G H.

Put i for the angle of obliquity, R and r for the two pitch radii A Q and B Q, N and n for the numbers of teeth in the two wheels, when, of course, $R : r :: N : n$, then $H G = (R + r) \sin i$; but the sum of the two circumferences is $2\pi (R + r) \cos i$, which must contain H G, the distance between two teeth, $t (N + n)$ times; therefore

$$(N + n) \tan i = 2 \pi t \qquad . \qquad (43.)$$

This equation establishes a connection between the number of teeth in contact at once, the angle of obliquity, and the number of teeth in both wheels. Thus, the number of teeth and the number of simultaneous actions being given, we can compute the angle of obliquity; but we are not certain whether such teeth will work, since the point of the tooth on one wheel may interfere with the consequent or antecedent tooth on the other wheel.

In order to avoid the complexity of duplicate surfaces, let us suppose the wheels to be destined for a movement in which the directions of the pressures are not to be reversed, and let us seek for the absolute limits of the teeth. The action of the tooth began when M reached H: as the motion proceeded, M entered within the circle $\beta' H \beta$, which, though defining the limit of pragmatic action, does not define the bottoms of the teeth. Describing upon the disc B a convoluted epicycloid, the point M only escapes all chance of interference with the antecedent tooth when it passes T, the intersection of the circles whose radii are A H and B G: similarly we must follow the trace made by the point N upon that part of the disc A which is included within the circle whose radius is A H, in order to ascertain whether there be any interference. These examinations can only be made in individual cases, and, to render the process obvious, we shall take as an example the following problem :—

To construct two wheels of 9 *and* 13 *involute teeth respectively, which may have the greatest possible number of simultaneous contacts.*

If the forms be determined for a pair of wheels on one scale of size, a proportionate enlargement or diminution will suffice to give the forms of wheels on any other scale: hence, instead of supposing the pitch radii A Q and B Q to be 9 and 13, we may, as in this investigation it is much more convenient, assume A G = 13, B H = 9, and change the distance A B according to circumstances.

Describe upon separate pieces of paper circles with the radii 13 and 9: divide their circumferences into 13 and 9 equal parts, and from each point of section spring an involute: the question is to determine the utmost lengths of these curves which can be used.

Beginning with the smallest possible number of contacts, let us take $t = 1$, then

$$\tan i = \frac{\pi}{11}; \; i = 15° .. 56' .. 21'', \text{ whence}$$

$$A Q = 13.520; \quad B Q = 9.360; \quad A B = 22.880;$$
$$A H = 14.439; \quad B G = 10.976; \quad H G = 6.283.$$

Having described on the discs circles with the radii A H and B G, we obtain the extents of the involutes α P M, β P N, as shewn in figure 42, Plate XXXII., and thence the convoluted epicycloidal arcs α N *s*, β M *t*, traced respectively by the points N and M. Here there is no danger of one tooth locking against another; indeed we might insert *one*, *two*, or even *three* teeth between a and α', without inconvenience; but in this way we would obtain wheels of 26 and 18 teeth with two, of 39 and 27 teeth with three, or of 52 and 36 teeth with four contacts. The backs of the teeth in these cases may be formed indifferently, provided the one do not interfere with the other; the epicycloids β M *t* and α N *s* shew the limits.

If we desire to increase the number of contacts without augmenting the number of teeth, we must give greater obliquity to the line H Q G. Assuming $t = 2$, we have

$$\tan i = \frac{2}{11}\pi,\ i = 29^\circ \,..\, 44' \,..\, 06'', \text{ whence}$$

$$AQ = 14.971\,;\quad BQ = 10.364;\quad AB = 25.336;$$
$$AH = 18.081\,;\quad BG = 15.457;\quad HG = 12.556.$$

Proceeding exactly as before, we obtain the results shewn in figure 43, Plate XXXIII., from which it is evident that the points of the teeth interfere with each other, the distance *s* M′ by which they overlap being an arc of 2° .. 22′. We cannot obtain two actions with wheels of 13 and 9 teeth.

By reducing the obliquity we may bring the point *s* just to M, and would then have the deepest possible tooth of this kind. Wheels constructed in this way cannot be used to drive backwards, and, as the enquiry is necessarily confined to one pair of wheels, the subject possesses little interest. When we take into consideration the back of the tooth, we are again brought to the truncated involute which has already been fully examined.

Teeth of the involute form possess, as is apparent from the above investigation, this very remarkable property, that the uniformity of the motion is not deranged by a change in the distance between the centres: and thus an error in placing the bushes does not produce any error in the working of the machinery. This property is characteristic of the involute, no other form of tooth enjoying it. Of this we may convince ourselves by resolving the following problem:

To construct wheels with teeth of such a form that their action may not be deranged by augmenting the distance between their centres.

Let C, figure 44, Plate XXIII., be the centre of a wheel of this kind, P the position of the point of contact, and P F part of the outline of a tooth.

In order to find the corresponding positions of the pitch points, we must draw at P a normal to the curve, and intersect this by circles described from C, with the pitch radii C Q, C Q′ corresponding to two assumed distances between the centres of the wheels. Now, it is an obvious property of two concentric circles, that the ratio of the cosines of the angles at which they cut any straight line is constant and inverse of the ratio of the radii. But, supposing for a moment the points C, Q, Q′ to be fixed, and the wheel to be turning, the velocity with which the contour at P retires from the pitch point Q must be to the velocity of the pitch line at Q as the sine of the angle C P Q is to radius; or, letting fall the perpendicular C R, as CR : C Q; and similarly for the point Q′. Hence, since the retirement from Q is just the same as that from Q′, if the wheels act alike for both pitch points, it follows that the velocity of retirement, the velocity of the pitch line at Q, and the velocity of the new pitch line at Q′, must be proportional to the three lines C R, C Q, C Q′. Thus these velocities belong to a disc turning on the centre, C. In this way, if we imagine some line C F fixed to the wheel, the velocity with which the angle Q C F increases in the one arrangement, must be exactly equal, throughout the whole extent of the motion, to that with which Q′ C F increases in the other arrangement: wherefore the variation of the angle Q C Q′ is, and always remains, zero; that is, Q C Q′ is a constant angle. But the lines C Q, C Q′ are also invariable, wherefore the trigon Q C Q′ is permanent. Now, in every position of the wheel, the normal drawn through P to the outline of the tooth must pass through Q; and, since the position of P is continually the same for both arrangements, the same normal must pass through Q′; wherefore it can be none other than Q Q′ itself. Thus the point of contact must move along the line Q′ Q P with a velocity equal to that of the disc at the distance C R, and must trace the involute of a circle of which C R is the radius. The involute, then, is the only form which permits a change in the distance between the centres of the wheels.

If we assume any other straight line for the path of the tracing-point, the motion along it must be unequable, and can afford us little assistance in shaping the tooth. The investigation of these cases then need hardly be undertaken except as geometrical curiosities. The most remarkable among them is that in which the tracing point describes a straight line parallel to the line of centres.

Having measured off any arbitrary distance Q F along the line Q E of figure 45, Plate XXXIV., draw through F a parallel to A Q B, then, according to what has already been shewn, if the motions be reckoned from the instant when the tracing point was at F, we have $FP^2 = 2q \cdot QF$, q being the distance passed over by the pitch lines, and the distance F P being laid on either side of F.

In order to obtain the form of the straight rack belonging to this system, measure off F E equal to Q, then E is the position which the vertex of the tooth occupies when the point of contact is at P, so that the tooth must have the form of a parabola, of which F Q is the focal distance. If we graduate the line F P to correspond with equal increments in the value of F E, and combine these successive positions of P with the graduations of a revolving disc, of which Q is the pitch point, we obtain the form of the tooth of a wheel. For the sake of illustration, one of these is represented in the figure.

We have now exhausted the case of rectilineal motion as far as it appears to bear on the practical part of our subject; let us proceed to consider the case of circular motion.

In order to find the law according to which the tracing point can describe the circumference of a circle, let F, figure 46, Plate XXXIV., be the centre, and F P the radius of the proposed orbit. Put F P = b, and denote the inclination of the radius F P to Q E by θ, then

$$x_P = x_F + b \cos \theta ; \; y_P = y_F + b \sin \theta ; \quad . \quad . \quad (44)$$

which, differentiated and combined with the condition of contact contained in equation (9), gives

$$\partial q = \frac{y_F \cos \theta - x_F \sin \theta}{x_F + b \cos \theta} \partial \theta, \quad . \quad . \quad (45)$$

the integral of which determines q in terms of θ. Actually performing the integration, we obtain

$$q = \theta y_F + x_F \log (x_F + b \cos \theta) - x_F y_F \int \frac{\partial \theta}{x_F + b \cos \theta}. \quad (45 . b).$$

This last integral has three distinct cases, according as b is less than, equal to, or greater than x_F; that is to say, according as the orbit lies all on one side of A B, touches A B, or intersects it.

In the first case the integral gives

$$q = y_F . \theta + x_F \log x - \frac{x_F y_F}{\sqrt{(x_F^2 - b^2)}} \cos^{-1} \frac{x_F \cos \theta + b}{x_F + b \cos \theta} + \text{C}. \quad (45 . c).$$

In figure 46, *a*, let F be the centre of the circular orbit, P the position of the tracing point. Draw F E parallel to A Q, and having joined F P, make p E s parallel to it, let fall the perpendicular Q s, and apply the tangent Q T to a circle described from E with the radius F P. Lastly, draw P p π parallel to A Q. Then the above equation may evidently be written thus—

$$q = \text{E F} \times \pi \text{E} p + \log \text{Q} \pi - \frac{\text{Q E . E F}}{\text{Q T}} \cos^{-1} \frac{ps}{\text{Q} \pi} + \text{constant.}$$

So long as F P is less than Q E, p s must be less than Q π, and the fraction $\frac{p s}{\text{Q} \pi}$, being less than unit, may always be the cosine of an angle, so that the tracing point may describe the entire circumference of the circle.

When the centre of the orbit is upon the line D Q E, the formula becomes remarkably simplified: In that case $y_F = 0$, wherefore

$$q = x_F \log x + \text{constant};$$

and if we reckon from the instant when P is on the line drawn through E, parallel to Q A, we obtain

$$q = x_{\text{P}} \log \frac{x}{x_{\text{P}}} \quad . \quad . \quad (45\ .\ d);$$

which is remarkable as being independent of the radius of the orbit: q is therefore the logarithm of the ratio Q π : Q E in the system of logarithms of which Q E is the modulus.

The entire progress of the tracing points may readily be exhibited thus. Let the contact begin at the lower vertex of the curve, that is when $x = x_{\text{P}} - b$, then

$$q' = x_{\text{P}} \text{ nep } \log \frac{x_{\text{P}} - b}{x_{\text{P}}};$$

and when the contact is about to cease we have

$$q'' = x_{\text{P}} \text{ nep } \log \frac{x_{\text{P}} + b}{x_{\text{P}}};$$

wherefore the length of the pitch line which runs out during one action is

$$\text{Q} = x_{\text{P}} \text{ nep } \log \frac{x_{\text{P}} + b}{x_{\text{P}} - b}.$$

Having computed this value of Q, and marked it off upon the pitch line of a disc, let it be divided into some number (say 40) of equal parts, divide the ratio Q E : Q e into the same number of equal ratios, and draw, through the extremities of the continued proportionals, lines parallel to B Q A, the intersections of these lines with the orbit will shew the successive positions of the two tracing points. As a matter of curiosity the forms of the rack-tooth, and of a wheel which makes one revolution during the passage of a tooth, are shewn in figure 46. b. From the nature of the genesis, it is easy to shew that the rack-tooth must always have the form of a meniscus, whose cusps correspond to the extremities of the diameter drawn through E.

In the second case, that is, when the orbit just touches the line of centres, the equation becomes

$$q = x_{\text{P}} \text{ nep } \log (1 + \cos \theta) + y_{\text{P}}\, \theta - y_{\text{P}} \tan \tfrac{1}{2}\, \theta; \quad . \quad (45\ .\ e)$$

and when the centre of the orbit is on the line D Q E, we have, as before,

$$q = x_{\text{P}} \text{ nep } \log \frac{x}{x_{\text{P}}};$$

but in this case the time during which the tracing point describes the entire circumference is infinite, since log. 0 = ∞.

When the orbit intersects the line A B, as happens when b is greater than x_{F}, the integral takes the form

$$q = x_{\text{F}} \text{ nep } \log x + y_{\text{F}}\, \theta - \frac{y_{\text{F}}\, x_{\text{F}}}{\sqrt{(b^2 - x_{\text{F}}^2)}} \log \frac{b + x_{\text{F}} \cos \theta + \sin \theta \sqrt{(b^2 - x_{\text{F}}^2)}}{x_{\text{F}} + b \cos \theta}.$$

This expression gives an infinite result for $x = 0$, and indicates that, in general, the tracing point cannot cross A B. However, the term involving the infinite logarithm has x_{F} for its co-efficient, and disappears when the centre of the orbit is taken in the line A B. In this case we have

$$q = y_{\text{F}} \cdot \theta, \qquad (46)$$

whence this remarkable property, that if from any point F in the line of centres, figure 47, Plate XXXVIII., a circle be described with any radius F P, the tracing point may move along the circumference of that circle with a uniform velocity, so as to make a revolution in the same time with a wheel of the pitch radius F Q. This is merely a case of our principal porism obtained by supposing the teeth of the wheel F to have collapsed into a point at P.

Excepting the wheel F, all wheels of this system have their teeth epicycloidal, the epicycloid being curtate when F P is greater than F Q, proper when F P is equal to F Q, and elongate when F P is less than F Q.

In order to place a circular orbit symmetrically around Q, its centre would need to be placed there, but then the velocity of the tracing point becomes infinite, for we have $y_{\text{F}} = 0$, whence $\partial q = 0 \times \partial \theta$.

If any valuable results be to be obtained from the circular orbit, it must be from the epicycloid proper, which brings the tracing point up to Q. Now this curve consists, like the involute, of two branches, one concave to the action, and the other convex to it, so that only one of those branches can have a mechanical existence. At each cusp one of the geometric contacts disappears, and must be replaced by introducing another epicycloid on the opposite side of the pitch line: that is by placing another circular orbit on the opposite side of Q, and making the tracing point leave the one circumference for the other alternately each time it comes to Q.

If the radii F Q and ϕ Q, figure 47, be equal to each other, the wheels formed in this way will be permutable: let us, then, examine closely the phases of the action.

To begin, as before, with the simplest case, let there be a single tracing point which, starting from Q, describes the circumference of the circle F, and, having again arrived at Q, leaves that circumference to describe the circle ϕ, and so on alternately; then, for each double revolution of the circle, F, the tracing point returns to its original state, and thus F Q must be half of the pitch radius of a wheel of one tooth, that is $\frac{1}{2}$ R of our preceding investigations. The outline of the straight rack becomes a succession of cycloids, arranged alternately to the right and left along the pitch line D Q E; and, being symmetrically placed, typifies a system of permutable wheels.

If we conceive a point to move along this contour, accompanied by a normal to the curve, the intersection of this normal with the straight pitch line will pass along the latter continuously with a variable velocity, but without any oscillation; therefore wheels of this system can never have more than a single point of contact, and the system is useless in a mechanical point of view. It may be noted that the depth of the tooth just reaches to the centre of a wheel of one tooth.

By enlarging the diameter of the generating circles, so as to make it some multiple of R, say the n^{th}, and by placing around each circumference n tracing points so disposed as to come alternately up to Q, we obtain a double series of cycloids intersecting each other in such a way that, by rejecting a trigonal space alternately on each side of the pitch line, we have a series of teeth satisfying, geometrically, the condition of permutability, and apparently having n points of contact. However, the entire number of geometrical contacts is curtailed by the excision of the middle part of each cycloid, and it even becomes a question whether the number of mechanical contacts be increased at all.

Having measured along the line D E, figure 48, the distances R S, S T, etc., each equal to $\frac{1}{2}\pi$, and, with some unknown distance ϱ for the radius of their generating circle, constructed the cycloidal arcs R X, T X, T Y, etc.: then, if a be the angular motion of that circle required for the description of the curve R X, we have

$$\mathrm{R\,S} = \varrho(a - \sin. a)\,;\quad \mathrm{S\,X} = \varrho(1 - \cos. a)\,;\quad . \qquad (47)$$

and if we measure off R q' equal in length to the arc ϱa, we obtain the normal X q'.

According to what has been already shewn, the number of pragmatic contacts is just equal to the number of times that R S is contained in R q'. Hence, if T be the number of pragmatic contacts, we have

$$a : \sin. a :: \mathrm{T} : \mathrm{T} - 1\,;\ \varrho = \frac{\pi\,\mathrm{T}}{2\,a} \qquad . \qquad (48).$$

The orbit of the tracing points in this case is composed of two circular arcs, and of their two chords, F G and H I; the tracing points for the plerotic contacts move with the same velocity as those for the pragmatic ones; also, since F Q G : F G :: T : T − 1, the plerotic are fewer by one than the pragmatic contacts. The analogy between this and the former orbits is apparent.

THE very same argument which we used in the case of the permutable involute may be employed here to shew that the teeth of all wheels in this system, and indeed in every system in which the rack-tooth is pointed, must be convoluted at the top; and we have to apply the same remedy, viz. the truncation of the teeth.

Figure 49, Plate XXXVII., representing a rack with truncated cycloidal teeth, put, as before, φ for the quantity of truncation Xx, so that R$s = \frac{1}{2}(\pi - \varphi)$, then we easily obtain

$$\frac{\sin a}{a} = \frac{\mathrm{T}-1}{\mathrm{T}} - \frac{\varphi}{\pi \mathrm{T}}\,; \quad \varrho = \frac{\pi \mathrm{T}}{2a}\,; \qquad (49)$$

by help of which any two of the three quantities φ, a, ϱ, can be found from the third.

Now, when φ is zero, the metacentre O, which marks complete pragmatic action, is at an infinite distance; and, as the value of φ is augmented, O approaches to Q, while ϱ, the radius of the orbit, evidently increases. Hence, since the metacentric distance Q N clearly coincides with ρ, the optimum arrangement must be to render the two metacentric distances, ρ and Q O, equal to each other. Describe then the arc $q'i$ equal to q'X, and join i and X with m, the centre of the orbit; lastly, with the radius mi describe the arc ib, then is ib one quarter of the path of the plerotic contacts. The time of describing bi must be to the time of describing q'X as T − 1 is to T, wherefore, since both motions accord with the revolution of a disc whose centre is m, the angles bmi and $q'm$X must be in the same ratio: now $q'm\mathrm{X} = a$, wherefore

$$b\,m\,i = a\,\frac{\mathrm{T}-1}{\mathrm{T}} \quad . \quad . \qquad (50).$$

This property may be easily represented geometrically. Having described a circle with any radius, $m\,q'$, figure 49, produce that radius until $q'm' = q'm$; make $q'm\mathrm{X} = a$, and join m'X, then ought the angles at m' and at m to be in the ratio of T − 1 to T. The point X is thus the intersection of a *magnetic curve* with the circumference of a circle. The exterior angle at the vertex X, is evidently $\frac{2\,\mathrm{T}-1}{\mathrm{T}}\,a$, wherefore

$$2 \sin\left(\frac{\mathrm{T}-1}{\mathrm{T}}\,a\right) = \sin\left(\frac{2\,\mathrm{T}-1}{\mathrm{T}}\,a\right) \quad . \qquad (51).$$

The determination of a by help of this equation is by no means tedious, particularly if attention be paid to the tabular differences. Thus, if we put $\mathrm{T}\lambda = a$, our equation becomes

$$2 \sin\,((\mathrm{T}-1)\,\lambda) = \sin\,((2\,\mathrm{T}-1)\,\lambda)\,;$$

and in the case of fifteen contacts, $T = 8$, whence

$$2 \sin 7\lambda = \sin 15\lambda.$$

Supposing that we have made the calculations of the preceding cases, it is easy to see that the value of λ must be about $2^\circ .. 45'$, hence, using tables to minutes,

$\lambda = 2^\circ .. 45'$	Log 2 = 0.301 0300	
$7\lambda = 19 .. 15$	L. sin = 9.518 1066	7 diff. = 2 5312
$15\lambda = 41 .. 15$	L. sin = 9.819 1133	15 diff. = 2 1600
	Error, − 233	3712

whence the correction of λ comes out $-3''.77$. Applying this correction, and using tables to seconds, we find the error to be zero: and if there had remained a small error, we should have proceeded in the same way. The values of λ and of a being thus obtained, it is easy to compute the dimensions of the orbits of the tracing points; these are given in Table IX. of the Appendix.

That form of the orbit which gives continually three points of contact is shewn in figure 50, Plate XXXVII., that for five contacts in figure 51, Plate XXXVIII., and that for seven in figure 52, Plate XXXIX., while figure 53, Plate XL., shews the rack engaged with a wheel of eleven teeth, which is the smallest wheel of this class that can have seven contacts.

A glance at the preceding tables serves to shew that teeth of this compound epicycloidal form possess an advantage both over the hour-glass odontoid and over the involute tooth in this respect, that, with a prescribed number of contacts, we can construct wheels of lower numbers: but this advantage is counterbalanced by an inconvenience that, just where the two epicycloids join on the pitch line, the radius of the curvature is zero, so that no cutter can ever form the outline truly: this inconvenience is, of itself, fatal to the use of the epicycloidal tooth. It is obvious also, from the figure, that the teeth of low-numbered wheels are undercut, so that they do not permit of the use of the rose cutter.

If, neglecting the condition of permutability, we compound the orbit of parts of two unequal circles, the epicycloidal tooth admits of an endless variety of cases. The only one of these which we shall notice is that case in which the diameter of each orbit is made equal to the pitch radius of the wheel on its own side. That part of the tooth of the wheel A, which is outside of its pitch line, has then the form of an external epicycloid, generated by rolling a circle whose radius is half of B Q round the pitch line whose radius is A Q: and conversely for the outer part of the tooth of the wheel B. The inner part of the outline of each tooth must be a radial line, since the epicycloid obtained by rolling a circle internally round another of twice its radius is the diameter of the stationary circle. If the external epicycloids be continued to meet each other, so as to form pointed teeth, the circle described by these points during the revolution of the wheel, gives the path of the plerotic contacts, and traces the bottom of the conjugate tooth. In this system each pair of wheels has to be separately considered. After the examples which we have already given, the reader will perceive at once the manner in which he may compute the number of contacts in any given case: the student can hardly do better than follow out the investigation as an exercise.

THROUGHOUT the foregoing investigations we have confined our attention to those orbits of the tracing point which cross the line D Q E only at Q. Now, although, from an augmented number of crossings, we can only expect increased complexity in the forms of the teeth, it is right, at any rate, that we should glance at the general effects which such an augmentation may have. The high importance of the subject requires that no department of its theory should be neglected since, even although no practical result may follow, we can hardly fail to have our ideas extended and generalised. To the speculative mathematician the enquiry presents attractions of no ordinary kind.

Assuming as indispensable the condition of permutability among the wheels, the orbit of the tracing point must be symmetrically placed around Q, and for this it is sufficient that only even powers of x and y enter into the equation of the curve: this equation may then be represented in general by

$$A + Bx^2 + Cy^2 + Dx^4 + Ex^2y^2 + Fy^4 + Gx^6 + \text{etc.} = 0. \quad (52).$$

THE first variety of this equation is obtained by stopping at the third term, and the curve so produced is the ellipse or hyperbola, according to the signs of the coefficients.

Let a be the semi-diameter of an ellipse in the direction D Q E, and b its conjugate semi-diameter in the direction A Q B, then the equation of the curve is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad . \quad (53)$$

which differentiated and combined with the condition of contact expressed in equation (9), gives

$$(a^2 - b^2)\, \partial x = a^2\, \partial q \quad . \quad (54).$$

The velocity of the tracing point measured in the direction x is thus constant. When the major axis of the ellipse lies in the direction D Q E, that is, when a is greater than b, the motion of the tracing point is upwards, and its velocity greater than that of the pitch lines in the ratio a^2 to $a^2 - b^2$. As a and b approach to equality, the velocity augments and becomes infinite when the ellipse merges into a circle. When a is less than b, the motion of the tracing points is downwards.

Integrating the last equation we have

$$(a^2 - b^2)\,x = a^2 q\,; \qquad . \qquad . \qquad (55)$$

hence if a straight line, always remaining parallel to A B, move upwards with the velocity $\frac{a^2}{a^2 - b^2}\partial q$, its intersections with the ellipse will shew the motions of the tracing points: now these intersections fail when x is beyond $\pm\, a$, and therefore the entire action is comprised between the limits

$$q = +\,\frac{a^2 - b^2}{a^2} \text{ and } q = -\,\frac{a^2 - b^2}{a^2}.$$

The form of the tooth of the straight rack is found by putting $x' = x - q$, whence

$$(a^2 - b^2)\,x' = b^2 q\,;\ \frac{a^2}{b^4}x'^2 + \frac{y^2}{b^2} = 1\,; \qquad . \qquad (56)$$

and thus the tooth of the straight rack is an ellipse, having the same breadth b with the orbit, but having its length a third proportional to a and b. Thus the forms of the orbit and of the tooth are interchangeable.

The shapes of the teeth of wheels may be deduced in the usual way; it is enough for us to observe that, in order to make the entire action last during the revolution of a wheel of one tooth, the radius of that wheel must be

$$\mathrm{R} = \frac{a^2 - b^2}{a\pi}.$$

Let D A E B, figure 54, Plate XLI., represent the elliptic orbit of the tracing points; make D Q : Q A : : Q A : Q e, then the ellipse d A e B is the form of the tooth of the straight rack; and e E is the distance through which the pitch lines move during the passage of the tracing point along A E; e E is therefore the semi-circumference of a circle whose radius is R = Q C′. Figure 55, Plate XLII., shews the form obtained by combining the motion of the tracing points in the orbit D A E B with the angular motion of a disc whose centre is at C′.

We propose, as an excellent exercise to the ambitious student, to determine, on the one hand, where C must be placed in order that the cusps may just meet, and, on the other hand, how remote it must be to cause the replication of the curve to disappear.

If *d* A *e* B be taken as the orbit of the tracing point, the ellipse D A E B becomes the tooth of the straight rack, while the distance *e* E still shews the extent of the motion of the pitch lines during the half transit of a tooth: but in this case the two tracing points appear simultaneously at *e*, and describe the arcs *e* A *d*, *e* B *d* to disappear together at *d*.

By changing one of the signs, equation 53 is transformed into that of an hyperbola, and by a similar proceeding we can shew that the velocity of the tracing points, estimated in the direction x, is uniform. There are two cases, according as the transverse axis of the curve is laid along A Q B or along D Q E; the investigation of the phases will present no obstacle to an assiduous student.

THIS variety of the general equation has a constant term, and the curve never passes through the point Q: and if we wish to restrict ourselves to lines passing through that point, we must omit the term A of the general equation. Again, the curve must not cross A Q B obliquely unless at Q, since the time of reaching any other crossing would be infinite. Now if, in the equation

$$Bx^2 + Cy^2 + Dx^4 + Ex^2y^2 + Fy^4 + Gx^6 + Hx^4y^2 + Ix^2y^4 + Ky^6 + \text{etc.} = 0,$$

we make x zero, in order to obtain the intersections of the curve with A Q B, we have, after rejecting the root $y^2 = 0$, which belongs to Q,

$$C + Fy_1^2 + Ky_1^4 + \text{etc.} = 0 : \qquad (57)$$

but $y \partial y$ ought also to be zero at the same time, and therefore,

$$B + Ey_1^2 + Iy_1^4 + \text{etc.} = 0 \quad . \qquad (58).$$

Every value of y_1 which satisfies equation (57) should also satisfy (58), and these two equations must be identic; in other words,

$$\frac{B}{C} = \frac{E}{F} = \frac{I}{K} = \text{etc.}$$

Again, if we desire to exclude all curves which cross D Q E perpendicularly, we make $y = 0$ in order to obtain the intersections with that line, whence dividing by x^2

$$B + Dx_1^2 + Gx_1^4 + \text{etc.} = 0 \quad . \qquad (59).$$

Now if any root of this equation should satisfy the condition $x \partial x = 0$, or

$$C + E x^2 + H x^4 + \text{etc.} = 0, \quad . \qquad (60)$$

the curve would cross D Q E at right angles, and therefore the two equations (59) and (60) must be incompatible.

From among the endless variety of cases which, with all these restrictions, may yet be proposed, we shall content ourselves with that which is typified by figure 5, where there are two perpendicular crossings of A B, and two oblique crossings of D E in addition to the crossing at Q.

In order that y_1 may have only one pair of values, equation (57) must be of the first order in reference to y^2, and similarly (59) must be simple in regard to x^2; wherefore our general equation is restricted to

$$B^2 x^2 + B C y^2 + B D x^4 + E y^2 (B x^2 + C y^2) = 0, \quad . \quad (61)$$

with the condition that E B be not equal to C D. And this we shall again restrict to the angular expression

$$x = A \sin 2\theta ; \; y = B \sin \theta + C \sin 3\theta ; \quad . \qquad (62)$$

as in this form the equation is best suited for our subsequent operations.

Combining the differentials of x and y with the condition of contact contained in equation (9), dividing all by sin θ cos θ, and then integrating, we obtain

$$2 (B^2 - 2 B C + 3 C^2)\theta + 4 (A^2 + 2 B C) \sin. 2\theta + 3 C^2 \sin. 4\theta = 4 A q. \quad (63)$$

When θ has passed through all values from 0 to 2π, these functions have returned to their original condition, and one tooth has completed its transit: hence, if we put R for the pitch radius of a wheel of one tooth, and α for the angular motion of that wheel, so that, as before, $q = R\alpha$, we have

$$\left.\begin{array}{c} B^2 - 2 B C + 3 C^2 = 2 A R \\ 4 (A^2 + 2 B C) \sin 2\theta + 3 C^2 \sin 4\theta = 4 A R (\alpha - \theta) \end{array}\right\} (64).$$

On attentively considering equations (62) it is seen that a change in the value of A merely compresses or elongates the curve, and that the relative values of B and C determine the character of the convolutions. When C is zero, we obtain the hour-glass curve of which we have already treated, and when B is taken zero we obtain a more complex knot, being one of the class of curves described by the free extremity of a straight vibrating wire of which the effective thickness is to the breadth as two to three; to this curve we shall give the name *kemend* (from كمند, Pers. a nooze).

In all other cases the characters of these two simple curves are commingled, so that, in studying these, our best course is to begin with the kemend, whose equation is

$$x = \text{A} \sin 2\,\theta;\; y = \text{C} \sin 3\,\theta \qquad . \qquad (65).$$

By merely putting zero for the B of the preceding equations, we find

$$3\,\text{C}^2 = 2\,\text{A}\,\text{R};\; 4\,\text{A}^2 \sin 2\,\theta + 3\,\text{C}^2 \sin 4\,\theta = 4\,\text{A}\,\text{R}\,(\alpha - \theta);$$

and if, as usual, we put $\text{R} = 1$, these become

$$3\,\text{C}^2 = 2\,\text{A};\; 2\,\text{A} \sin 2\,\theta + \sin 4\,\theta = 2\,\alpha - 2\,\theta; \quad . \quad (66)$$

which again may be put under the more convenient form

$$(\text{A} + \cos 2\,\theta) \sin 2\,\theta = \alpha - \theta.$$

Whatever be the ratio of A to C there are always four points, F, G, H, I, figure 56, at a maximum distance from Q, so that a circle described from Q with the radius Q F would just contain the whole curve. When A is great in proportion to C, these points, along with K and L, are the only places at which normals will fall from Q: but when A is short, the kemend is compressed, and a normal may fall upon each of the arcs F K, K G, H L, L I. Now the foot of each normal, excepting those which happen on the line of centres, indicates the appearance or disappearance of a pair of contacts, and therefore when A is long we have four, when A is short eight points, at which the contacts begin or cease. In the latter case, that part of the curve on each side of K or L, which is included between the new normals, is described backwards.

To find the limit which separates these two cases we observe that, if N be the intersection of any normal with the line of centres,

$$6\,\text{C}\,.\,\text{Q}\,\text{N}\,.\cos 3\,\theta = 2\,\text{A}^2 \sin 4\,\theta + 3\,\text{C}^2 \sin 6\,\theta; \qquad (67)$$

and therefore the points at which normals fall from Q upon the curve are to be determined by resolving the equation $\text{Q}\,\text{N} = 0$, or

$$\sin 2\,\theta\,\{\,12\,\text{C}\,(\cos 2\,\theta)^2 + 4\,\text{A}^2 \cos 2\,\theta - 3\,\text{C}^2\,\} = 0.$$

The root $\sin.\ 2\,\theta = 0$ of this equation corresponds to the two points K and L: the other solution

$$-\cos 2\,\theta_1 = \frac{\text{A}^2 \mp \sqrt{(9\,\text{C}^4 + \text{A}^4)}}{6\,\text{C}^2} \qquad . \qquad (68)$$

determines the position of the eight oblique normals. The upper alternative of this is always possible, and gives the points F, G, H, I; but the under alternative fails just when

$$6\,C^2 = A^2 + \sqrt{(9\,C^4 + A^4)} \text{ or } 3\,C = 2\,A,$$

and at this limit the circle which osculates the curve at K has its centre at Q. Observing that when $R = 1$, $2\,A = 3\,C^2$, we find that the limit of continuous progression upwards is when $C = 1$, $A = 1\frac{1}{2}$, and then the velocity of the tracing point at K is infinite. As was the case with the hour-glass curve at the analogous limit, the kemend intersects the axis at Q at an angle of 45°.

In order that the number of contacts be constant, the time in which the tracing-point describes the arc Q G, must be some multiple, as the T$^{\text{th}}$, of the quarter time in which a tooth passes; that is to say, the value of θ_1 in equation (68), must give $\alpha_1 = \frac{\pi}{2}\,T$.

Combining equations (68) and (66) so as to eliminate first C and then A, we obtain

$$\sin\,4\,\theta_1 - 2\,\frac{\sin\,6\,\theta_1\,.\,\sin\,2\,\theta_1}{\sin\,4\,\theta_1} + 2\,\theta_1 = \pi\,T; \qquad (69)$$

from which the form of the orbit necessary to secure constantly a given number of contacts may be deduced.

On applying this equation to the case $T = 2$, that is, to the case of three contacts, we obtain

$$\theta_1 = 39^{\circ}\,..\,37'\,..\,27''.93\;;\quad A = 2.307248\;;\quad C = 1.240228\;;$$

and thereafter, by help of equation (66), the values of x and y set down in Table X. of the Appendix. From these values of x and y, the path of the tracing points, as shewn in figure 56, has been protracted. Combining the motions therein shewn with that of a straight rack, there results the figure shewn in 57 *a*, Plate XLIV.; of this outline two parts are very nearly straight, representing, as it were, the straight sides of figures 32 and 35; but here the truncation is effected by scooping out the top of the tooth, and by inserting a rounded protuberance in the bottom. Figure 57. *b*, Plate XLV., shews the form of the wheel of one tooth. As was to have been expected from the first, this system of teeth is utterly impracticable; but by combining the kemend with the hour-glass curve, we may modify the regions of plerotic contact; and it is a proper subject for investigation, whether thereby some advantages may not be obtainable, particularly in regard to the positions of the metacentres.

HAVING formed an hour-glass curve, figure 58, Plate XLVI., with the dimensions $x = \mathrm{A} \sin 2\theta$, $y = \mathrm{B} \sin \theta$, if we annex to the value of y a term of the form $\mathrm{C} \sin 3\theta$, we shall occasion a modification greater or less according to the value of the coefficient C. For the value $\theta = 60°$ the term $\mathrm{C} \sin 3\theta$ is zero independently of C; and therefore, all the modifications must intersect the principal curve and each other at the nodes k, l, m, n, corresponding to $\theta = 60°$, $\theta = 120°$, $\theta = 240°$, $\theta = 300°$.

Supposing C to be positive, let it increase gradually from zero: the branch Q G k will be thrown forwards, and the part k B l inwards, quickening the turn at the shoulder of the curve, and flattening the summit of the tooth. The shoulder of the curve at last becomes sharp, as shewn in figure 59, presenting the appearance of a cusp: this discriminating phase of the curve occurs when $\mathrm{C} = + \frac{1}{3}\mathrm{B}$. It would be an interesting enquiry, how the coalescence of the contacts at such a cusp would affect the form of the tooth.

If C continue to increase, loops are formed at the four corners, and present, when $\mathrm{C} = + \frac{2}{3}\mathrm{B}$, the appearance shewn in figure 60. Should C still increase, the vertex of the curve approaches to Q, and reaches that point when $\mathrm{C} = + \mathrm{B}$; this phase is shewn in figure 51: after this the curve assumes the general aspect of the kemend.

Supposing on the other hand that C, being taken in the negative direction, gradually increases from zero, the curve will be drawn inwards along the arc Q G k, and protruded along k B l, as shewn in figure 58. When — C becomes $\frac{1}{3}$ B, the tangent of the inclination at Q is zero, that is to say the two branches of the curve cross and osculate each other at Q, as in figure 59; presenting there in some measure the aspect of the epicycloidal orbit. Beyond this value of — C the tangent of the inclination becomes negative, and the curve presents the general appearance of figure 5.

It is obvious at a glance, that all practicable modifications of the hour-glass curve are included witbin the limits

$$x = \mathrm{A} \sin 2\theta; \; y = \mathrm{B} \sin \theta \pm \tfrac{1}{3}\mathrm{B} \sin 3\theta.$$

Although it would be highly interesting to examine minutely the phases which accompany each of these limits, it is more consistent with the general tenor of our enquiry, to seek within them for a curve possessing some real advantage over those which we have already studied.

When a small positive value is ascribed to C, the vertex of the curve at A of figure 58 is flattened, and the radius of curvature increased, so that the metacentric distance Q M of figure 13 being augmented, more pinions are thrown out of use. But on attributing to C a small negative value, the vertex of the curve is made rounder, the centre of curvature is brought nearer to Q, and wheels of smaller numbers become practicable: at the same time the tendency is to augment the metacentric distance Q N. We have already shewn that, in all practicable forms of the hour-glass curve, Q M is greater than Q N, hence the effect of attributing a small negative value to C is to bring the two metacentres together, and, if we desire to have the complete number of contacts plerotic as well as pragmatic, the optimum modification is obtained by giving C such a value as to make the two points M and N coalesce.

In this case the convolutions will disappear simultaneously from the sides and from the point of the tooth. Let us endeavour to investigate this modification. For this purpose, and in order to follow the direct algebraic course, we may put $C = -\gamma B$, γ being a fraction less than $\frac{1}{3}$: this, along with the supposition $R = 1$, gives

$$\left.\begin{aligned} 2A &= B^2 (1 + 2\gamma + 3\gamma^2); \\ x = \tfrac{1}{2} B^2 (1 + 2\gamma + 3\gamma^2) \sin 2\theta&;\ y = B(\sin \theta - \gamma \sin 3\theta); \end{aligned}\right\} \qquad (70)$$

now the general value of $Q n$ is $y + x \frac{\partial x}{\partial y}$, whence

$$Q n = B\{(1 - 3\gamma) \sin \theta + 4\gamma \sin \theta^3\} + B^2 (1 + 2\gamma + 3\gamma^2) \frac{\sin \theta - 2 \sin \theta^3}{1 - 3\gamma + 12\gamma \sin \theta^2}. \quad (71)$$

Putting in this expression $\sin \theta = -1$, in order to obtain the radius of curvature at the vertex, we find

$$Q M = B^2 \frac{(1 + 2\gamma + 3\gamma^2)^2}{1 + 9\gamma} - B(1 + \gamma). \qquad . \qquad (72).$$

Let h be the value of θ corresponding to the maximum value of $Q n$, then, equating the differential coefficient of (71) to zero, we obtain

$$0 = \{1 - 3\gamma + 12\gamma \sin h^2\}^3 + B^2\{1 + 2\gamma + 3\gamma^2\}^2\{1 - 3\gamma - 6(1 - \gamma)\sin h^2 - 24\gamma \sin h^4\} (73).$$

Having obtained from this equation, which is of the third degree in reference to $\sin h^2$, the value of $\sin h$, and inserted it for $\sin \theta$ in (71), we would have

$$Q N = B \sin h \left\{ 1 - 3\gamma + 4\gamma \sin h^2 + B^2 (1 + 2\gamma + 3\gamma^2) \frac{1 - 2 \sin h^2}{1 - 3\gamma + 12\gamma \sin h^2} \right\}; (74)$$

our condition of optimism requires that Q M be equal to Q N, and hence, if we were to eliminate $\sin h$ from equation (73) and from $Q N = Q M$, we should obtain the law of connection between B and γ necessary to bring the two metacentres together.

Again, in order to determine the number of simultaneous contacts, we must ascertain the value of θ corresponding to the point G; denote this value by ψ, then, since the normal must pass through Q. we have

$$0 = 1 - 3\gamma + 4\gamma \sin \psi^2 + B^2 (1 + 2\gamma + 3\gamma^2) \frac{1 - 2 \sin \psi^2}{1 - 3\gamma + 12\gamma \sin \psi^2}, \quad . \quad (75)$$

which is only quadratic in respect of sin ψ^2.

Inserting the value of ψ thus found in the general equation of motion (64), and observing that in this case $\alpha = \frac{1}{2}\pi$ T, T being the number of pragmatic contacts, we have

$$(\pi T - 2\psi)(1 + 2\gamma + 3\gamma^2) = \{B^2(1 + 2\gamma + 3\gamma^2) + 8\gamma\} \sin 2\psi + 3\gamma^2 \sin 4\psi \ (76).$$

If we could eliminate ψ from equations (75) and (76) we would obtain another law of connection between B and γ necessary to give 2 T — 1 simultaneous contacts. And, by help of these two laws we could finally compute B, γ, and every other dimension of the orbit. A single glance at the enormous complexity of these equations is sufficient to extinguish all hope of succeeding by this direct process, and therefore we must endeavour to reach the results by some smoother, though circuitous road.

As the system of seven contacts offers, of all others, the greatest practical advantages, let it be proposed to ascertain the dimensions of the modified hour-glass curve which shall give seven contacts, and whose metacentres M and N shall coincide. For this purpose we shall set aside the condition R = 1, and, in its stead, use a fixed value for A until we have obtained a satisfactory result: after that we can easily restore the proper value to R by changing the scale of the orbit. By this means we avoid unnecessary divisions. Selecting 5 as a convenient value for A, and reducing the dimensions of the hour-glass curve for seven contacts in the ratio of 5 to 5.475, we obtain

A = 5.000, B = 3.022, C = 0.000, $(\alpha) = 360°$.
R = 0.913, Q M = 30.068, Q N = 10.267.

As a first modification, let us preserve the above values of A and B, putting C = .200 : we then find at once R = 1.046; Q M = 17.516 : it remains for us to compute the values of Q N and of (α). Resolving the cubic equation

$$0 = \{(B+3C)^3 - 8A^2(B+3C)\} + \{-18C(B+3C)^2 + 12A^2(B+3C)\} \cos 2h$$
$$+ \{108C^2(B+3C) - 24A^2C\} (\cos 2h)^2 - 216C^3 (\cos 2h)^3,$$

we obtain cos $2h$ = .703 4240, whence Q N = 10.7294795. And again, resolving the equations,

$$0 = B^2 + 2BC - 3C^2 + \{4A^2 - 8BC\} \cos 2\psi + 12C^2 (\cos 2\psi)^2$$

$$(\alpha) = \psi + \frac{4A^2 - 8BC}{4AR} \sin 2\psi + \frac{3C^2}{4AR} \sin 4\psi,$$

we find the angular motion of the wheel of one tooth to be $(\alpha) = 307^\circ .. 06' .. 39''$, instead of the 360° necessary to ensure always four pragmatic contacts.

On comparing these with the preceding results, it appears that we must augment a little the ratio of C to B, as also that of A to B; we can accomplish this by leaving to A and C their present values, and by assigning a smaller value to B.

Let us then make a second attempt by putting $A = 5.000$; $B = 2.900$; $C = 0.200$. Proceeding exactly as before, we find $R = 0.969$, $QM = 18.176\,600$, $QN = 11.130\,655$, $(\alpha) = 328^\circ .. 18' .. 43''$.

These results guide us to another trial, almost at hazard, with $A = 5.000$, $B = 2.800$, $C = 0.180$; which give $R = 0.89452$, $QM = 19,644430$, $QN = 11.409251$, $(\alpha) = 353^\circ .. 37' .. 33''.3$.

Applying to these three rude approximations the well-known formulæ for the variations of binome functions, we obtain a fourth approximate case, the results of which, contrasted with the preceding, give a still closer approximation.

Proceeding in this very laborious course, we at last find for $A = 5.000$, $B = 2.49132516$, $C = 0.36906730$, the values $R = 0.84542664$, $QM = 14.342633$, $QN = 14.342626$, and $(\alpha) = 360^\circ .. 00' .. 00''.40$, where the discrepancies are within the limits of the errors of our seven place logarithmic tables: we have also, at the same time,

$$2h = 37^\circ .. 58' .. 53''.86, \text{ and } 2\psi = 94^\circ .. 44' .. 07''.86.$$

In order to accommodate these dimensions to our general formulæ, we must have $R = 1$, which is obtained by dividing all the linear values by 0.8454 2664, whence, ultimately,

A	= 5.914 1737	Log. A	= 0.771 8941
B	= 2.946 8259	Log. B	= 0.469 3545
C	= 0.436 5456	Log. C	= 9.640 0296
Q M	= 16.964 9651	Log. Q M	= 1.229 5530

wherefore the equation of contact becomes

$$\alpha = \theta + \sin\, 2\theta \,.\, \log^{-1} 0.738\,7127 + \sin\, 4\theta \,.\, \log^{-1} 8.383\,2265,$$

by help of which the values of θ, and thence those of x and y, corresponding to each division of the passage of a tooth, have been computed and recorded in Table XI. From these values the forms of the orbit shewn in figure 62, Plate XLVIII., and of a rack engaged with a wheel of 17 teeth in figure 63, Plate XLIX., have been constructed.

If we were to attempt the construction of wheels belonging to this system with fewer than 17 teeth, replications would make their appearance at the points and also at the shoulders of the teeth; so that, although the replication at the point might not include any of the pragmatic contacts, such wheels would be impracticable on account of the hiatus in the action elsewhere. Were it not, then, that we come within the metacentric distance $Q\,N$, we might reduce the number of teeth until the centre of the wheel should arrive at the metacentre O : on making the computation, we find $Q\,O = 9.283959$, and hence the question naturally suggests itself—would it not have been preferable, instead of bringing up the value of $Q\,N$ to that of $Q\,M$, to have augmented it only to meet that of $Q\,O$?

For the purpose of answering this question, we must replace the condition $Q\,N = Q\,M$, by this other $Q\,N = Q\,O$, and make our computations over again : but since the calculation of the value of $Q\,O$ is much more tedious than that of $Q\,M$, the labour of this new operation must be vastly greater than that of the preceding.

After a long series of extremely tedious approximations, we obtain

$$\begin{array}{lll}
A & = 5.730\,568 & \text{Log. } A = 0.758\,1977 \\
B & = 3.137\,134 & \text{Log. } B = 0.496\,5331 \\
C & = 0.232\,316 & \text{Log. } C = 9.366\,0792 \\
Q\,O & = 13.466\,450 & \text{Log. } Q\,O = 1.129\,2531 \\
Q\,M & = 21.756\,440 & \text{Log. } Q\,M = 1.337\,5879
\end{array}$$

$2\,\psi = 95^{\circ}..\,05'..\,33''.22$. From these results it appears that the smallest pinion of this system must have fourteen leaves, in order that the pragmatic action may be complete; and that the smallest wheel must have twenty-two teeth at and beyond which the replications disappear from the points of the teeth, and the plerotic contacts also are complete.

The condition of contact becomes

$$\alpha = \theta + \sin\,2\,\theta\,.\,\log^{-1} 0.738\,4802 + \sin\,4\,\theta\,.\,\log^{-1} 7.849\,0222,$$

by help of which the co-ordinates of the tracing points, as entered in Table XII., have been computed. From these the form of the orbit, and the motions of the tracing points shewn in figure 64, Plate L. have been laid down, and the outlines of the straight rack and of the pinion of 14 leaves, given in figure 65, Plate LI., have been protracted.

Of the five permutable systems which we have examined, the last seems to be the most advantageous on this account, that, with seven contacts, it admits of pinions so low numbered as fourteen. It is true that the truncated epicycloidal system goes down even to eleven teeth, but these cannot be opened by the rose cutter, neither can the very sudden change of curvature at the shoulder of the tooth ever be correctly given. We may, therefore, finally assume the hour-glass curve modified to suit the condition $QN = QO$ as the best type for our system of wheels.

OUR investigations hitherto have had reference only to the conditions of contact, of permutability, and of continuity in the substance of the wheels: there are two other considerations of great importance; viz. the amount of friction and the manner in which the parts wear. These we shall now proceed to examine, premising that, however lengthened and tedious the preceding enquiries may appear to have been, they scarcely form more than an introduction even to that department of the general subject.

In order to compute the loss of force caused by the friction of the surfaces, we must ascertain the distance through which the one tooth slides over the other. Returning to figure 1, Pp is the element of the outline of the one tooth, and $P\varpi$ that of the outline of the other tooth, wherefore $p\varpi$ is the distance through which the sliding takes place; in fact, the points p and ϖ simultaneously arrive at P', and therefore the intervening surfaces must have slid over each other. Since the trigons AQP' and BQP' are similar to $P'rp$, and $P'r\varpi$, we have

$$rp = \frac{QP' \,.\, P'r}{AQ}\,;\ r\varpi = \frac{QP' \,.\, P'r}{BQ}\,;\ \text{whence,}$$

$$\varpi p = QP' \,.\, P'r \left\{ \frac{1}{AQ} \mp \frac{1}{BQ} \right\}$$

the sign — being used when the centres are on the same side of Q, and the sign + in the more usual case, when the centres are on opposite sides of Q. Or, observing that, in equations (4) and (5) $R v \partial t = -\partial q$; $r \varphi \partial t = -\partial q$, and subtracting, we obtain

$$p\varpi = \partial q \left\{ \frac{1}{R} - \frac{1}{r} \right\} \sqrt{\left\{ x_P^2 + y_P^2 \right\}} \quad . \quad . \quad . \quad (77)$$

which agrees exactly with the geometrical result.

This expression for the amount of sliding is the product of two factors, of which the one depends on the radii of the two wheels, and is constant for them, and of which the other $QP \,.\, \partial q$ depends entirely on the nature of the orbit assigned to the tracing point.

Put W for the resistance to be overcome, estimated at the circumference of the pitch circle, and C for the co-efficient of friction belonging to the specific materials of which the wheels are constructed; then W sec ω, that is, W sec $QP'R$ is, according to the laws of Statics, the strain which must be exerted in the direction Q P, or the pressure of the one surface against the other. Multiplying this by the co-efficient of friction, and by the distance through which the sliding takes place, we obtain the element of the loss of force occasioned by friction. Putting F for this loss of force, we have

$$\partial F = W C \left\{ \frac{1}{R} - \frac{1}{r} \right\} Q P . \sec \omega . \partial q \quad . \quad . \quad (78).$$

Of this expression the part Q P . sec ω . ∂q alone depends on the form of the tooth : for our purpose, therefore, we may put

$$F = W C \left\{ \frac{1}{R} - \frac{1}{r} \right\} f,$$

in order to obtain the simpler equation

$$\left.\begin{aligned} \partial f &= Q P . \sec \omega \, \partial q = \frac{Q P^2}{x_P{}^2} \partial . Q P \\ &= \frac{x^2 + y^2}{x^2} \{x \partial x + y \partial y\} \end{aligned}\right\} \quad (79).$$

This combined with the equation of the orbit, and then integrated, gives us the entire loss of force accruing from the motion of the wheels through a given distance; and if we observe that W q is the amount of force transmitted after deduction of the friction, the quotient by W q will give us the rate of loss.

We may observe, as a general rule for the application of this equation, that the integral ought to be restricted to the pragmatic part of the orbit, that is, it ought to be taken from Q P = 0 to the first occurring maximum value of Q P. The plerotic part of the orbit serves to indicate the limit beyond which the solid matter of the wheel ought not to project, but below which it may be awanting without detriment to the dynamic action: only, since by excavating below this limit we impair unnecessarily the strength of the tooth, we ought to keep to it as nearly as possible.

When the tracing point describes a straight line drawn through Q at an inclination i to D Q E, the value of f is at once found to be

$$\left.\begin{aligned} f &= \frac{1}{2} \sec i^2 . Q P^2 = \frac{1}{2} q^2 \\ F &= \frac{1}{2} W C \left\{ \frac{1}{R} - \frac{1}{r} \right\} q^2, \end{aligned}\right\} \quad (80)$$

whence this very remarkable property, that the loss of force arising from the friction of involute teeth is independent of the obliquity of the action, and is proportional simply to the square of the distance passed over by the pitch line. An increased obliquity augments the pressure on the surface of the tooth, but lessens in the same ratio the amount of sliding. Since the force transmitted is Wq, the proportion which the loss bears to the effect is

$$\frac{1}{2}\,C\left\{\frac{1}{R}-\frac{1}{r}\right\}q = \text{relative loss}, \qquad . \qquad (81)$$

for teeth of the involute form.

In order to compute the friction on teeth belonging to the hour-glass curve system, we must substitute in equation (79), X sin 2θ for x and Y sin θ for y, whence

$$\partial f=\left\{X^2 \sin 4\theta+\frac{3}{2}Y^2 \sin 2\theta+\frac{Y^2}{4X^2}(Y^2-4X^2)\tan\theta\right\}\partial\theta$$

which integrated and reckoned from $\theta = 0$ becomes

$$f=\frac{1}{2}x^2+\frac{3}{2}y^2+\frac{Y^2}{4X^2}(Y^2-4X^2)\text{ nep log sec }\theta, \qquad . \qquad (82\,a)$$

or, observing that, in order to simplify our calculations, we have usually made $R = 1$, which, in the case of the hour-glass curve, gives $Y^2 = 2X$,

$$\left.\begin{aligned} f &= \tfrac{1}{2}x^2+\tfrac{3}{2}y^2+(1-2X)\text{ nep log sec }\theta \\ &= \tfrac{1}{2}X^2(\sin 2\theta)^2+3X(\sin\theta)^2+(1-2X)\log\sec\theta. \end{aligned}\right\}\ (82\,b).$$

The value of this has to be taken for the whole extent of the pragmatic action, that is, from $2\theta = 0$ to $2\theta = 90° + \psi$.

If we attempt to apply this formula to the plerotic part of the orbit, that is, from $\theta = 45° + \frac{1}{2}\psi$ to $\theta = 90°$, the result comes out infinitely negative. In truth, the sliding changes its direction as soon as we pass into the region of plerotic contact, and the integral, if reckoned onwards from $\theta = 0$, having reached its maximum value, begins to decrease, thus presenting one of the numerous cases in which the usually received manipulation of the signs + and — fails us.

For the friction of epicycloidal teeth, we put ρ for the radius of the orbit, so that

$$x=\varrho\sin\frac{q}{\varrho};\ y=\varrho\text{ ver }\frac{q}{\varrho};\ \omega=\frac{q}{2\varrho},\ \text{whence}$$

$$f=4\varrho^2\log\sec\omega;\ c\left\{\frac{1}{R}-\frac{1}{r}\right\}\frac{2\varrho}{\omega}\log\sec\omega=\text{relative loss } (83).$$

Let us now investigate the loss of force in the systems belonging to the modified hour-glass curve. Here we have $x = A \sin 2\theta$, $y = B \sin \theta - C \sin 3\theta$, whence

$$4\,A^2 \partial f = \{\,(B+C)^2 + (4\,A^2 - 8\,B\,C - 8\,C^2)\cos\theta^2 + 16\,C^2\cos\theta^4\}\tan\theta\,.\,\partial\theta$$

$$\times\{-4\,A^2 + B^2 + 10\,BC + 9C^2 + (8A^2 - 16\,BC - 48C^2)\cos\theta^2 + 48C^2\cos\theta^4\}$$

Performing this multiplication and integrating, observing also to reckon from $\theta = 0$, we find

$$4A^2 f = \{-4\,A^2 + (B+C)(B+9C)\}\,(B+C)^2 \log\sec\theta$$

$$+\ \{3A^2B^2 + 6A^2BC + 4A^2C^2 - 6B^3C - 18\,B^2C^2 - 20\,BC^3 - 6C^4\}\ \text{vers}\ 2\theta$$

$$+\ \{A^4 - 4\,A^2BC + 6\,B^2C^2 + 4\,BC^3 + 3\,C^4\}\ \text{vers}\ 4\theta$$

$$+\ \left\{A^2C^2 - 2\,BC^3\right\}\frac{5}{3}\ \text{vers}\ 6\,\theta$$

$$+\ \left\{\frac{3}{4}C^4\right\}\text{vers}\ 8\,\theta\,.$$

By help of these formulæ we are able to compute the friction in wheels belonging to any of the five systems of which we have treated. In every case the differential of the friction is a product of that of Q P, and therefore, provided none of the factors be infinite, these two differentials are zero together. Now, the only factor which, in practicable cases, can become infinite, is x^{-2}; therefore, whenever QP happens to have a maximum value out of the line of centres, the friction f attains also a maximum value. That is to say, proceeding from Q to I on any orbit, the quantity of friction goes on increasing until it have reached its maximum at I, thereafter the algebraic integral decreases; but, in reality, we have no such thing as negative friction, and thus in computing actually the friction belonging to any part of the orbit which includes I as a I *b*, figure 13, we must integrate from a to I and from I to *b*, and add the two results together, irrespective of their signs. For the reasons which have already been given, the friction belonging to the arc IB need not be taken into consideration.

When there is only one point of contact, the process which we have followed gives us the loss of force occasioned by the friction; but, when there are several points in action at once, it is questionable whether the same rule hold good.

The entire pressure transmitted from the one wheel to the other is divided among the points of contact, but is not necessarily shared equally amongst them: a very slight irregularity in the surface is sufficient to throw the whole strain upon a single point. Let us suppose that, of two wheels perfectly formed and placed at the proper distance, one is securely fixed, and the other left free to turn on its centre: while no strain is applied, the surfaces touch at several points; but, as soon as we apply a pressure tending to turn the free wheel round, it advances, compressing the solid matter near certain of the contacts, until the repulsions thus excited be sufficient to balance the angular strain. If the curvatures at those points of contact were all alike, the resistances would be equal to each other; but, in all practical cases, the curvatures are unequal, and a greater repulsion is evoked at those contacts where the curvatures nearly fit than at those places where the surfaces separate suddenly.

Had the law according to which the curvatures regulate the repulsion, been known, we could have investigated the share of the strain apportioned to each contact. No attempt, however, has yet, as far as I know, been made to elucidate this class of phenomena, so that we are obliged to fall back on the simple supposition that the torsion is equally distributed among the effective contacts.

When the entire number of contacts is of the form $4n - 1$, that of the pragmatic contacts is $2n$, of which n are forwards and n backwards, so that the number of acting contacts is constantly n. In this case, just when one contact is ceasing at G, another is preparing to supply its place at H; and it may be observed, that for all analytic curves, or rather, for all orbits which are not cornered at G, F, H, I, the surfaces recede from or approach to each other gradually: in this way the distribution of the strain is not disturbed. But when the entire number of contacts is of the form $4n + 1$, there are $2n + 1$ of them pragmatic, and of these $n + 1$ may be forwards and n backwards, or n may be forwards and $n + 1$ backwards. Thus, when a contact ceases at G, its place is supplied by another at F, which, however, is of no use in resisting the actual pressure: thus the entire torsion has sometimes to be borne by $n + 1$, sometimes by n points, according to the value of α. Each disappearance of a contact at G is thus accompanied by an increase of compression at each of the remaining points, which increase continues until the reappearance of a contact at H. However carefully the teeth may be formed, there will thus be a shock or tremor at each change of contact; so that truly smooth and continuous action can only be obtained when the entire number of contacts is of the form $4n - 1$, as 3, 7, 11, etc., and when the parts Q G and F G of the orbit glide into each other at G.

When the number of pragmatic contacts is odd, the friction is distributed unequally on different parts of the orbit; thus, in figure 14, the part H Q G being that in which the working contacts occur, when a contact has just appeared at H, there is another at 10 on Q G; these move simultaneously along the orbit agreeably to the divisions shewn in the figure, and reach the points numbered 30 at the same instant: during this period each supports one-half of the strain. The contact now ceases at G, and the whole strain is thrown upon the single contact over the part 30 Q 10. In this way, while the wheels have moved forward one tooth, we have half the friction over the whole arc H Q G, together with the half friction over 30 Q 10, or otherwise the sum of the two frictions over Q G and Q 10.

Again, in figure 19, which shews the motions of nine contacts, when a contact has just appeared at H, there are others at the two points, 30, 30, on H Q G; these move simultaneously into the positions 10, 10, 10, each bearing one-third part of the strain: at that instant one contact ceases at G, and the remaining two contacts move simultaneously from 10 to 30, each bearing one-half of the strain: thus the entire friction during the passage of a tooth is made up of one-third of the friction over the three parts 30, 10, and one-half of the friction over the two parts 10, 30. Similarly we can ascertain the distribution of the friction in any other case where the entire number of contacts is of the form $4n + 1$.

In the case of seven contacts, figure 17, the arcs 0, Q and Q, 40 are described by the two acting points of contact during the passage of a tooth, and therefore the friction during that time is just the friction on Q G.

In the case of eleven contacts, the three parts, 20, 20, 20, 20, are each described during the transit of a tooth, and therefore the entire friction during that transit is two-thirds of what belongs to the arc Q G : and in general, when the number of contacts is $4n - 1$, the friction during the passage of a tooth is two n^{th} parts of what would belong to the arc Q G, described by a single contact bearing the full strain.

On applying these formulæ to the different cases, we arrive at the results which are entered in the subjoined Table :—

$$\text{Relative Loss} = C\left\{\frac{1}{R} - \frac{1}{r}\right\} \times \text{coefficient.}$$

Contacts.	Hour-Glass.	Involute.	Epicycloid.	Q N = Q M.	Q N = Q O.
3	1.603 103	1.570 796	1.648 290		
7	3.151 349	3.141 593	3.189 000	3.161 547	3.165 506
11	4.715 194	4.712 389	4.753 851		
15	6.308 014	6.283 185	6.322 170		

It thus appears that teeth of the involute form have the advantage over all the others in respect of friction; but the differences are too small to be of any practical importance. The second modification of the hour-glass curve, which in other respects is the most advantageous, is, in this respect, the least so; but the difference only amounts to three-fourths of a per cent. Roughly it may be remarked that the friction is proportional to the number of teeth fully engaged.

As an example, let it be required to determine the loss occasioned by the friction of two iron wheels of the numbers 16 and 57 belonging to the system last investigated.

The friction of iron on iron may be taken at one-tenth of the pressure; hence, when the contact is external, and the radii have different signs, the formula becomes

$$\frac{1}{10}\left\{\frac{1}{16}+\frac{1}{57}\right\} \times 3.165\,509 = 0.025\,338,$$

and thus the loss of force amounts to about two and a-half per cent. In the case of internal contact the loss is only 0.014 231.

To compute the loss of force in a train of wheels, put ϕ', ϕ'', ϕ''', etc. for the relative losses at the several actions computed by the above formula; then if a force 1 be communicated to the last wheel, the preceding one must have exerted a force of $1 + \phi'$; and again for each unit of force exerted by the penult wheel, there must have been $1 + \phi''$ exerted by the ante-penult, so that to produce unit on the last wheel, we must have had a force of $(1 + \phi')(1 + \phi'')$ in action at the ante-penult wheel:—thus the force at the extremity of the train being unit, that exerted by the driver must be

$$(1 + \phi')(1 + \phi'')(1 + \phi''') \ldots \text{etc.},$$

that is, if the friction on the pivots be not taken into account.

The loss of force caused by the attrition of the surfaces of the teeth is accompanied by a wearing of those surfaces. In the absence of precise experiments to determine the laws of this class of phenomena, we may concede that the entire quantity of matter abraded is proportional to the force expended in friction: if this be so, it must be advantageous to use wheels as large as circumstances will allow, since thereby not only is the friction reduced, but the abrasion is distributed over a larger surface, and therefore the deterioration lessened in a still higher ratio.

It is a very important question, in what way does the abrasion affect the form of the tooth? or, to put it in another light, *what is that form of tooth which shall not be deranged by the wearing of the parts?*

Referring again to figure 1, we observe that the sliding πp takes place while the contact shifts from P to p on the tooth of the wheel A, and from P to π on the tooth of the wheel B. Hence the abrasion due to this quantity of sliding is distributed over Pp on the contour of the one wheel, and over Pπ on that of the other.

If we may be allowed to suppose that the quantity of matter abraided is proportional to the extent of the sliding, and to the intensity of the pressure, the depths of the abrasions will be inversely as Pp and Pπ, hence that depth, measured in the direction P$'$ Q perpendicular to the surface, will be

$$e\frac{\partial q}{\mathrm{P}p}\left\{\frac{1}{\mathrm{R}}-\frac{1}{r}\right\} \mathrm{QP} \,.\, \sec \mathrm{QPR}$$

for the wheel A, e being some unknown constant multiplier. In order to find the angular displacement, we must augment this in the ratio of $s\,P'$ to $P'\,p$, and divide the result by the distance $A\,P'$. Observing that $A\,P'$. sin. $A\,P'\,Q = A\,Q$. cos $Q\,P'\,R$, we find for the amount of angular displacement

$$\frac{e}{R}\left\{\frac{1}{R} - \frac{1}{r}\right\}\frac{Q\,P}{P\,p}\sec i^2\,\partial q; \quad . \quad (85)$$

and, in order that the form of the tooth may remain unaltered, the value of this expression ought to be the same along the whole side of the tooth; that is to say, the expression

$$\frac{Q\,q}{P\,p}\,Q\,P\,.\sec i^2 = a, \quad . \quad . \quad (86)$$

as well as the corresponding one for the wheel B, viz.:

$$\frac{Q\,q}{P\,\pi}\,Q\,P\,.\sec i^2 = b, \quad . \quad . \quad (86)$$

ought to be constant throughout the entire action.

Supposing that we have found an orbit which satisfies this condition for the wheel A, it is requisite, in order that the two wheels may continue to work together, that the same orbit satisfy also the same condition for the wheel B. Hence, if such an orbit be possible, we must have $a\,.\,P\,p = b\,.\,P\,\pi$, that is to say, the elements of the two contours must be in a constant ratio. Since the three points P, p, π, are in one straight line, this equation may be resolved into these two

$$\left.\begin{aligned} a\,(x_p - x_P) &= b\,(x_\pi - x_P) \\ a\,(y_p - y_P) &= b\,(y_\pi - y_P) \end{aligned}\right\} \quad . \quad . \quad (87).$$

Substituting these in equations (4) and (5), so as to eliminate ∂q, we find after simplification,

$$\left\{a - b - \left(\frac{a}{R} - \frac{b}{r}\right)y\right\}\partial y = \left\{\frac{a}{R} - \frac{b}{r}\right\}x\,\partial x,$$

which gives, on being integrated,

$$x^2 = 2\,y\,R\,r\,\frac{a - b}{a\,r - b\,R} - y^2, \quad . \quad . \quad . \quad (88)$$

which is the equation of a circle whose centre is in the line A Q B, and whose circumference passes through Q. Thus the contours of the two teeth must be epicycloids produced by rolling this circle upon the two pitch lines. This result might easily have been anticipated by those conversant with the properties of the epicycloid.

Thus, if it be possible to construct wheels whose teeth shall not wear out of shape, these teeth must be of the epicycloidal form. We have already seen that the simple epicycloid is inadmissible in practice: but to complete the investigation, we may enquire whether the condition (86) be compatible with this form of tooth. Putting ϱ for the radius of the orbit, we have $QP = 2\,\varrho, \sin i$, whence

$$\frac{\varrho}{b} = \cos i^2 - \frac{\varrho}{r} \sin i \,.\, \cos i,$$

an equation of which one side is constant and the other variable, and which, therefore, represents an unattainable condition. Hence we conclude that it is impossible to construct wheels such that the form of their teeth may not be deranged by wearing,—admitting, however, the law of abrasion as above enunciated: but this law is purely hypothetical, and does not contain all the conditions of the action, for the curvatures of the parts, which curvatures may be expected to have a very important influence on the wearing, have not been at all considered. Enough has been said on this head to shew that a wide and unexplored field for experimental research lies open before us

In the general equation with which we set out, the co-ordinate z is indeterminate, so that we were enabled to simplify our formulæ by supposing z to be constant. Thus our attention has all along been restricted to a section of the wheel-work made by a plane perpendicular to the axes. Since a point in this section is never, by the rotation of the machinery, carried out of it, each section by a plane parallel to X Y, is independent of every other section; and thus we are at liberty to make all the sections equal to each other, or to vary them in any way that may be expedient, provided the laws already pointed out be fulfilled in each individual section.

The usual arrangement is to make, throughout the whole thickness of the wheel, all the sections equal and parallel to each other. This, besides being the readiest, is decidedly the most advantageous form, since the normal to the surface of the tooth is always perpendicular to the axis of the wheel. Whenever the normal is inclined to the plane X Y, the pressure is augmented in the ratio of the secant of the angle of obliquity to radius, and the friction also in the same proportion. The pressure, in this case, may be decomposed into two, one in the plane of the motion, which is what is necessary to propel the machinery, and one proportional to the tangent of the obliquity, perpendicular to the former, and which causes an unnecessary pressure of the axes end-ways against the sides of their boxes. When the obliquity is zero, the secant is at its minimum, and the tangent is zero, so that, of all possible arrangements, the common one is the best.

It may happen, however, that other arrangements become advisable, and therefore it is proper that we should not altogether pass them over without notice.

In order to give a greater degree of accuracy to the motion, a plan has been adopted of dividing the thickness of the wheel into strata placed in succession a step back. By this means the strain is kept in the plane of the motion, and the advantage is sought of combining a greater number of contacts: the tracing points are thus distributed at smaller intervals over the orbit, and there results a kind of compensation of errors. It is precisely as if a series of wheels, all cut alike, were placed side by side on the same axis, but successively displaced so as to divide the interval between two teeth into as many parts as there are wheels. The adjustment of the wheels on the one axis to those on the other, is, however, a matter of great delicacy, and it comes to be questionable whether any advantage result, or rather whether it would not have been better to have bestowed this additional care on the original formation of the tooth. When the steps are to be elaborated in one piece of metal, the difficulty of reaching their surfaces must enhance vastly the amount of labour, and cause it far to exceed what would be required to obtain the same degree of precision in the ordinary way.

When the arguments for the position of the tracing point depend on the ordinate z, the normal to the surface of the tooth will, in general, be inclined to the plane X Y, and the teeth may assume an endless variety of forms, each depending on the peculiar nature of the connection between z, y, x, and q. Should q alone be a function of z, the q of one section will differ, by a constant quantity, from the q of another, but the forms of the sections will be alike, only they will be displaced by the constant difference, in such a way that, if the relation between q and z be of the form of $q = m + n z$, the surfaces of the teeth will be helical. If, on the other hand, the dimensions of the orbit vary with z, the forms of the sections will vary also, and we may conceive teeth of one class gradually changing into teeth of another class, so as to present, on the same wheel, all the intermediate gradations.

However interesting these phases may be to the speculative mathematician, they possess little or no attraction to the practical inquirer: yet there is one case of this gradual change of form which we shall afterwards be led to examine, when treating of skewed gear.

In the preceding parts of this Treatise we have considered those principles which ought to regulate the formation of the teeth of wheels, and have computed the dimensions of the orbits of the tracing points with much more precision than can be attained in practice. Thus on wheels whose teeth are separated to the distance of 2π, or upwards of six inches, our tables give the form of the tooth to within the ten thousandth part of an inch: now on a well constructed planing engine, or on a good lathe, it is barely possible, trusting to the divisions of the leading screws, to come within the thousandth of an inch, so that the tabulated decimals go beyond what can ever be required. On teeth of smaller pitch the precision is still greater.

It is now proper to examine the effects of those imperfections which are unavoidable, and which arise principally from two sources; inequality in the graduation of the index-plate, and inexactitude in the shaping of the cutter.

In the present state of machine making, errors in the index-plate ought never to be more than microscopic, and unequally divided *cut* wheels ought never to be spoken of, unless when they are employed to regulate the movement of some delicate astronomical instrument. It is only in wheels *off the sand* that inequalities in the distances of the teeth can be tolerated. If two irregularly divided wheels be brought into gear, a thin tooth in the one may come up to a wide space in the other, and there will result a slackness, or *back-lash*, as it is named: but, again, a thick tooth may come up to a narrow interval, in which case the motion will be arrested: for this reason it is necessary to pitch the distance between the centres, so as to allow of a slight shake, by way of avoiding the risk of the wheels being locked. When we consider that the necessity for this back-lash arises from errors in the dividing plate, or from bad centering the wheels, we conclude that, in well-arranged machinery, it ought scarcely to be perceptible, and we regard the smallness of its amount as a kind of index to the good quality of the workmanship.

Errors of the other class, those arising from malformation of the cutter, are much more difficult to be avoided.

If we bring the teeth of two wheels, A and B, to touch each other, note the point of contact P, figure 67, Plate LII., and there raise a normal to the two surfaces, this normal will cut the line of centres at some point R; then it is easy to see that, if the positions of the wheels be slightly disturbed, their angular velocities will be inversely as the distances A R, B R; in other words, R is the pitch point for the moment. When the teeth are properly formed, R remains stationary, and coincides with the position of Q as deduced from the numbers of the teeth in the two wheels.

If R remain stationary during the action of one pair of surfaces, yet not at Q, the angular motions of the two wheels during this time must be disproportioned to the numbers of the teeth; and, therefore, at the escape of each tooth there must occur a drop backwards of the wheel which has been propelled too far; or, if the propeller have advanced beyond its proper place, its next tooth must catch abruptly the succeeding tooth in the other wheel. The same thing will happen if the position of R, though variable, be confined to one side of Q. When wheels with this defect are used in cranes, we hear a distinct blow at each interval of a tooth, or the heavers experience a sudden increase of strain according as R lies on the side of the wheel or on the side of the pinion. Should R oscillate on either side of Q, the effects of the malformation will be less perceptible, though not the less real, and by accident it may happen that the excess of motion while R is on the one side may be just balanced by the deficiency while R is on the other side of Q.

The errors in the teeth of clock-work are principally evinced by variations in the arc of vibration of the pendulum, arising from changes in the intensity of the pressure on the pallets. The effect of a single blow is imperceptible, and it is only by the accumulation of many inaccuracies that any obvious change can be produced; hence it is that the influence of irregularities in the swift moving wheels is scarcely perceptible, and that it is only from the teeth of the great wheels that any evil can arise. In all cases the variations are periodic, and the disturbance of the *mean rate* of the time-keeper is zero; and thus even a shockingly ill-constructed clock will keep good time, provided its pendulum-rod be made of a material whose thermal expansion is small.

In watch-work the friction on the pivots of the balance is much greater in proportion to the quantity of force, and the impulse which must counteract this friction is also proportionally greater: hence the influence of irregularities in the pressure is much more rapidly developed in watch than in clock-work. To obviate this defect, the contrivances called *remontoir* are introduced, However, as the oscillation of a pendulum and the vibration of a watch-balance, though not perfectly isochronous, are very nearly so, the influence of the imperfect formation of the teeth is minute, and it is only in astronomical time-keepers that any serious inconvenience is felt.

It is when wheel-work is the vehicle for conveying *force* that the evil effects of badly-formed teeth are fully seen, for then the periodic changes in the relative velocities develope pressures which augment with the speed of the machinery, and which may become immensely greater than that which the teeth were intended to exert.

Let us suppose, for the purpose of fully explaining our ideas, that the shaft of the wheel B is attached to some enormously heavy machinery revolving steadily with a determinate velocity, while the shaft of the wheel A carries a fly.

When the teeth of both wheels are truly formed, the fly revolves uniformly on its axis, and no pressure is exerted by the teeth of the wheel B upon those of the wheel A, beyond what is necessary to overcome the friction on the axis and the resistance of the air. These, which are foreign to our present subject, may be neglected; or, to give the matter a more practical aspect, we may suppose that a pressure just sufficient to meet the friction, and the air's resistance, is applied directly to the shaft of the wheel A, say by means of a driving belt. In this state of affairs there will be absolutely no pressure between the teeth, and the two wheels will go on revolving together, as if independent. It is only while the speed of the machinery is being changed that any pressure will be exerted, and as soon as that change is effected, and the velocity is again made uniform, the pressure ceases. During acceleration B will urge A forwards, and during retardation A will tend to drag B forwards, the intensity of the pressure depending on the rapidity with which the change of speed is made.

When the teeth are badly shaped, it is quite another matter, for then the two wheels cannot move uniformly, or, if the motion of one of them, as B, be supposed uniform, that of the other must be variable, and the variations must be periodically reproduced at each passage of a tooth. Put F for the weight of the fly-wheel, and all that is attached to the same shaft, and v for the linear velocity which the mean circumference of gyration would have had if simply annexed to the axis of B, then its velocity in its actual position must be

$$v\,\frac{BQ}{AQ} = v\,\frac{r}{R}$$

(putting r for B Q, and R for A Q) if the teeth be truly formed. To find its actual velocity at any time, let s denote the distance between the true pitch point Q, and the instantaneous one R counted towards A, then we have

$$V = v\,\frac{r+s}{R-s}$$

for the variable velocity. According to the laws of phoronomics, the *quantity of force* residing in the fly-wheel at this instant is

$$\phi = \frac{F}{2g}\,v^2\,(r+s)^2\,(R-s)^{-2} \quad . \qquad (90.)$$

When the point R, and consequently the value of s, changes, the quantity of force ϕ changes also; but this change can only be produced by the application of a pressure acting through some distance in such a way that if p denote the pressure, and δq. the distance through which it has acted, we must have $p\,\delta q = \delta\,\phi$. Now x is a function of q, determined by the shape of the tooth, wherefore

$$p = \frac{F}{g}\,v^2\,(R+r)\,(r+s)\,(R-s)^{-3}\,\frac{\partial s}{\partial q} \quad . \qquad (91.)$$

In well made machinery, where the error s results from those minute inaccuracies which are unavoidable in our most carefully conducted operations, the factor $(r + s)\,(R-s)^{-3}$ will be very nearly $r\,.\,R^{-3}$, so that

$$p = \frac{F}{g} \frac{R+r}{R} \frac{r}{R^2} v^2 \frac{\partial s}{\partial q}$$

$$= \frac{F}{g} \frac{R+r}{Rr} V^2 \frac{\partial s}{\partial q}; \qquad . \qquad (91 \,.\, b).$$

and thus it appears that the pressure arising from the imperfection in the shape of the teeth, is proportional to the velocity with which the point R shifts its position, and to the square of the velocity of the machinery. Thus a mere augmentation of velocity, though unaccompanied by any increase in the inciting pressure, may develope strains sufficient to endanger the machinery.

When the point R remains stationary on one side of Q during the action of a pair of surfaces, the pressure on those surfaces is zero, but the instant they cease to act, the succeeding teeth strike each other with a blow, the weight of which depends on the distance through which R leaps. The pressures resulting from this blow are determined by the nature of the material, the masses of the wheels, and the elasticities of the shafts which connect them with their flies; that is, in general, on the amount of *yielding* of the parts. In all cases it is vastly greater than any pressure which could result from a graduated motion of the point R. When the burden is suddenly removed from a train of such wheels, and the speed in consequence becomes greatly accelerated, it now and then happens that the teeth are broken off, their strength, though quite adequate to sustain the usual resistance, being insufficient to encounter the pressures evoked by their own irregularities.

These considerations suggest a very simple and effectual process for finishing the surfaces of teeth which have been carefully cut.

Let the wheels be attached to two axes on which are fixed, at alternate ends, two fly-wheels: then having brought them as close as they can work, and communicated a motion to one of them by means of a long strap, let an appropriate grinding powder be strewed upon them: by this means the projecting parts of the teeth will be abraded and their formation improved. The motion ought to be frequently reversed, and the wheels now and then turned face for face, in order to obtain a variety of combinations, and a slight end-shake on one of the shafts should be encouraged, by way of preventing ridges. As the process proceeds, the wheels should be brought closer, finer powder should be used, and the speed accelerated until the required degree of smoothness be attained.

ON THE COMPUTATION OF TRAINS OF WHEELS.

HAVING now considered very fully those principles which regulate the forms of the teeth of wheels, we have to treat of those which determine the selection of particular numbers. Our general object in employing toothed wheels is to establish a proportion between the revolutions of two axes: that proportion being determined by the nature and uses of the machinery of which the axes are to form part.

This, which at first glance seems a very simple department of our subject, is often surrounded with grave difficulties. The principal source of these difficulties is the necessity of operating with wheels of a moderate number of teeth, which is forced upon us by the limits to the strength of the material with which we have to operate.

Thus, if we attempt to connect the minute-wheel of a clock directly with the day-wheel, we must use numbers in the ratio of 1 to 720. Now the smallest available number of teeth with good action is 14, so that we should have a wheel of 10 080 teeth driving a pinion of 14 leaves. It is evident that such a high-numbered wheel is quite impracticable: on a diameter of 10 inches the distance between the two teeth would be less than the 320th part of an inch. It is thus practically impossible to obtain such a high ratio by one action: we must have recourse to a series of actions, technically called a *train*. In ordinary clocks the movement is arranged in some such way as this: a pinion of 8, fixed on the minute axis, is driven by a wheel of 60, fixed on an intermediate axis, which therefore turns *backwards* in 7½ minutes. Upon this axis a pinion, say of 9, is fixed, which pinion is propelled by a wheel of 72 teeth, making a turn *forwards* in one hour. Upon the hour axis there may be fixed a pinion of 20, leading a wheel of 80, and upon this wheel a pinion of 25 geared into a wheel of 75; this last will then turn once in twelve hours. In this way, by help of moderately-sized wheels, we obtain the desired result.

From this example of a simple train, we see that each intermediate axis carries a pinion and a wheel, but that the extreme axes carry the one a pinion, the other a wheel only. It is also apparent that the axes turn alternately to the right and left, we must use internal contact when we wish to make two consecutive axes turn in the same direction. Hence the action of the series may be represented thus,

$$\frac{-8}{60} \times \frac{-9}{72} \times \frac{-20}{80} \times \frac{-25}{75} = \frac{+1}{720}.$$

Generally, if the driving wheels have a, b, c, d, etc. teeth, while the driven ones have A, B, C, D, etc. teeth, the ratio of the ultimate angular velocities will be

$$\frac{-a}{A} \cdot \frac{-b}{B} \cdot \frac{-c}{C} \cdot \frac{-d}{D} \text{ etc.}$$

the sign of the result indicating whether the extreme axes turn in the same or in opposite directions.

When the numbers of the teeth are given it is thus very easy to compute the resulting ratio, but it is by no means such a simple matter to find a train which may give a prescribed ratio. Let n : N be the desired ultimate ratio expressed in numbers prime to each other, then we must resolve these into their factors, so as to have

$$\frac{a}{A} \cdot \frac{b}{B} \cdot \frac{c}{C} \text{ etc.} = \frac{n}{N},$$

and when one of these numbers has fewer prime factors than the other, unit may be inserted as a factor to supply the place; thus in the above example,

$$\frac{1}{720} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{5} \cdot$$

When the factors are small we may group them in any way that may be convenient, and we may also multiply any numerator, provided we multiply some denominator by the same number; thus

$$\frac{1}{720} = \frac{1}{15} \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{3} = \frac{2}{15} \cdot \frac{1}{8} \cdot \frac{1}{4} \cdot \frac{1}{3},$$

which last is used in the train noticed above. But pinions of one leaf are impracticable, we therefore multiply the numerator and denominator of each fraction by any convenient number, taking care that none of the products be too large.

Should any of the prime divisors of n and N be very large, the solution of the problem is impossible by a simple train such as we have been considering, and we must have recourse to some other contrivance. But, for the great mass of practical purposes, it is sufficient to approximate closely to the prescribed ratio, the degree of approximation being suited to the object proposed to be obtained.

It is in the construction of orreries and other instruments for representing the motions of the heavenly bodies that we find the most beautiful examples of this kind of calculation. We shall select one of these by way of illustration.

Thus, let it be proposed to connect the hour-wheel of a clock with an axis which shall turn once in the mean tropical year. According to the determination of Professor Bessel, the duration of the tropical year is 365.242 217 mean solar days, and the probability of error may be taken at about the half of unit in the sixth decimal place. If, then, we can obtain a series of wheels which give a result between 365.242 2165 and 365.242 2175, our representation will possess all the precision which can be attained in the present state of astronomical science.

Our business is now this; to find two numbers n and N each decomposable into moderate factors, and such that the ratio n : N may give the length of the tropical year within the above prescribed limits. This problem belongs to the *Theory of Numbers*, and particularly to that branch of it which treats of the properties and management of Brounker's fractions. On this subject the reader may find advantage in consulting "*Legendre, Essai sur la Théorie des Nombres, Paris, An. VI.* (1798), or *Barlow's Theory of Numbers, London*, 1811. The problem, in its present shape is, I believe, new; and, as the line of investigation which we are about to follow is also peculiar, I shall enter considerably into detail.

In the first place, we assert that, though restricted to wheels below some fixed limit as 300, 400, etc., it is always *possible* to construct a simple train which shall give a proposed ratio within any required degree of exactitude.

Let α and β be two given prime numbers, and l some ratio which it is proposed to obtain by means of a train of wheels into which the numbers α and β alone enter effectively. We have to resolve the indeterminate equation

$$\frac{\alpha^m}{\beta^n} = l$$

either exactly or very nearly. Taking the logarithms of both sides, we obtain

$$m \,.\, \text{Log } \alpha - n \,.\, \text{Log } \beta = \text{Log } l, \quad . \quad (92)$$

which resembles an indeterminate equation of the first degree, except in this, that the three logarithms are incommensurable.

Proceeding according to the well known process to seek, as if it were possible to find, the common measure of log α and log β, we obtain a series of quotients, which we may denote by a, b, c, etc.; and by help of these we construct a series of Brounkerian fractions,

$$\frac{1}{0}, \ \frac{a}{1}, \ \frac{ab+1}{b}, \ \frac{abc+a+c}{bc+1}, \text{ etc.}$$

of which the values, alternately in excess and defect, approach with great rapidity to the value of the ratio log α to log β.

Let $\frac{p}{q}$ be an odd, $\frac{r}{s}$ an even term of this series, then the quantities

$$p \text{ Log } \beta - q \text{ Log } \alpha \text{ and } r \text{ Log } \alpha - s \text{ Log } \beta$$

are always positive, their values decreasing as the fractions are taken farther along in the series; so that, by proceeding far enough, these may be rendered less than any assignable quantity. In other words,

$$\frac{\beta^p}{\alpha^q} \text{ and } \frac{\alpha^r}{\beta^s}$$

may be made to exceed unit by an error which may be rendered as minute as may be required.

Having subtracted log α as often as possible, say Q times, from log l, and obtained a residue which, for the sake of uniformity, we shall denote by log l', so that

$$Q \log \alpha + \log l' = \log l,$$

let us compute the value of the first quantity of the series $r \log \alpha - s \log \beta$. Should that quantity be less than log l', let it be subtracted as often as possible, say R times, leaving a second residue log l'', then

$$(Q + Rr) \log \alpha - Rs \log \beta + \log l'' = \log l.$$

Here the error is reduced to log l'', which is necessarily less than $r \log \alpha - s \log \beta$.

Computing now the value of the next expression, $r' \log \alpha - s' \log \beta$, and subtracting it as often as possible, S times, from log l'', we obtain

$$(Q + Rr + Sr') \log \alpha - (Rs + Ss') \log \beta + \log l''' = \log l$$

where the error is still smaller: and by continuing this process sufficiently far, we may obtain

$$m \log \alpha - n \log \beta = \log l$$

with any required degree of approximation.

When restricted to the use of two prime numbers, it is always possible to resolve the indeterminate equation $\alpha^a . \beta^b = l$, where a and b are integers, which may be taken with either sign, to the required degree of exactitude: much more then, when we are allowed the use of all prime numbers, α, β, γ, δ, etc. under 400 or 500, must it be possible to satisfy the equation

$$\alpha^a . \beta^b . \gamma^c . \delta^d . \text{etc.} = l$$

to the same degree of precision. Hence, however tedious the actual computation may turn out to be, we are cheered in the labour by the certainty that a practical result is always *possible*,—unless indeed the train become too cumbrous from its length.

Returning to the original form of our problem, we observe that, in order to its resolution, we must be able to obtain all rational fractions whose values are included within the two given limits, and to decompose the members of those fractions into their prime factors. Amongst them, there must of necessity be some, all of whose factors are below the largest number prescribed for our wheels: Our business is to render the search for these perfectly effective and as expeditious as possible.

Let l and L be two ratios differing little from each other, it is proposed to indicate the form of all rational fractions intermediate between them.

In seeking to represent l by a Brounker's fraction, we obtain a series of quotients, a, b, c, etc. by help of which, following the method of Legendre, we easily compose a series of fractions, alternately too great and too little, approximating to the value of l. Again, for L we obtain another set of quotients and a corresponding series of fractions. When l and L differ but slightly from each other, several of the quotients, a', b', c', will be equal to a, b, c, and the two series of approximating fractions will only begin to diverge after having coincided for some time. Let $\frac{p}{q}$ and $\frac{r}{s}$ be the last two fractions which are common to both series, and suppose that e, f, g, etc. e', f', g', etc. are the subsequent quotients, then for the one series we shall have

$$l <> \frac{p}{q}, \frac{r}{s}, \frac{er+p}{eq+s}, \frac{(ef+1)r+fp}{(ef+1)s+fq)}, \text{ etc.}$$

and for the other series,

$$\mathrm{L} <> \frac{p}{q}, \frac{r}{s}, \frac{e'r+p}{e'q+s}, \frac{(e'f+1')\,r+f'p}{(e'f'+1)r+f'q}, \text{ etc.}$$

so that, if $\frac{\mathrm{E}}{\mathrm{F}}$ be the resultant of the quotients e, f, g, etc., and $\frac{\mathrm{E}'}{\mathrm{F}'}$ that of the quotients e', f', g', etc., we shall have

$$l = \frac{\mathrm{E}r+\mathrm{F}p}{\mathrm{E}s+\mathrm{F}q}, \ \mathrm{L} = \frac{\mathrm{E}'r+\mathrm{F}'p}{\mathrm{E}'s+\mathrm{F}'q}; \quad . \quad . \quad (93)$$

now it is an obvious property of any two contiguous terms as $\frac{p}{q}, \frac{r}{s}$, that all fractions intermediate between them must be of the form $\frac{m\,r+n\,p}{m\,s+n\,q}$ where m and n are two integer numbers, of which m is not the lesser. Therefore, *a fortiori*, all fractions between the narrower limits l and L must be of the same form; but not conversely. In order to restrict ourselves within these limits, we must assume m and n, so that the fraction $\frac{m}{n}$ be intermediate between $\frac{\mathrm{E}}{\mathrm{F}}$ and $\frac{\mathrm{E}'}{\mathrm{F}'}$; that is, between $\frac{p-lq}{ls-r}$ and $\frac{p-\mathrm{L}q}{\mathrm{L}s-r}$. (94).

The discovery, then, of all rational fractions between the limits l and L, requires the indication of all those between the limits $\frac{\mathrm{E}}{\mathrm{F}}$ and $\frac{\mathrm{E}'}{\mathrm{F}'}$. Thus we are brought to a problem exactly of the same nature with that from which we set out. There is, however, this difference that the interval between the new limits is great, and that it includes at least one integer number, as is evident from the circumstance that e' differs from e.

It is most convenient to compute the values of the limits (94) in decimals: then having put $n = 1$, we make m equal to each of the included integer numbers: next we put $n = 2$, and for m each integer between the doubles of the limits, omitting, however, the even numbers, as these would reproduce the fractions previously obtained: again, we put $n = 3$, and for m each integer not divisible by 3, which is found between the triples of the limits, and so on.

This is a part of the investigation which presents no practical difficulty: the real labour is in discovering the prime divisors of the members of the several fractions thus found. That most excellent work of Burkhardt, his *Table des Diviseurs, Paris* 1817, affords every facility for this research, so long as the numbers do not exceed 3036 000: for numbers beyond that we must recur to the usual laborious method of trial.

In order to obtain the divisors of a large number, we have to attempt the division of it by each prime less than its square root. Having obtained one divisor, we treat the quotient in the same way, omitting those primes already tried unsuccessfully, and continue this process until we obtain a quotient known to be prime. For our present purpose, it is sufficient to try all primes under the extreme limit which has been assigned to the number of teeth in one wheel. The primes 2, 3, 5, offer no difficulty, neither do 7, 11, 13, as their continued product 1001 is easily managed; but it is another affair with the higher primes. The discussion of the most economical way of conducting these trials belongs properly to a treatise on numbers: it is sufficient that we point out one artifice which, in general, serves to reduce the labour within bearable limits.

N being some large number whose divisors are sought, let P be the continued product of many primes, α, β, γ, etc.; then, if N be divisible by any of these primes, P and N must have a common divisor; hence, if we seek for the common measure of N and P we shall have effected at once all the trials by α, β, γ, etc. Also, whenever the remainder is brought below 3 036 000, its divisors may be found by help of Burkhardt's Table; and if any of these be among α, β, γ, etc. that one may be a divisor of N. For facilitating this operation, the following products are subjoined:—

$$17 . 19 . 23 . 29 . 31 . 37 . 41 . 43 . 47 . 53 . 59 . 61 . 67 . 71 = 18\ 579\ 448\,222\ 667\,298\ 067\ 513.$$

$$73 . 79 . 83 . 89 . 97 . 101 . 103 . 107 . 109 . 113 . 127 . 131 = 942\ 566\ 757\,362\ 434\,851\ 686\,917.$$

$$137 . 139 . 149 . 151 . 157 . 163 . 167 . 173 . 179 . 181 . 191 = 1\ 960\ 258\ 687\ 761\ 048\,923\ 918\,053.$$

$$193 . 197 . 199 . 211 . 223 . 227 . 229 . 233 . 239 . 241 . 251 = 62\ 340\,438\ 891\,512\ 738\,845\ 905\,557.$$

$$257 . 263 . 269 . 271 . 277 . 281 . 283 . 293 . 307 . 311 . 313 = 950\ 372\,124\ 774\,798\ 979\,851\ 590\,827.$$

Should N itself happen to be a divisor of P, it must be the continued product of some of the primes α, β, γ, and there is no alternative but to try them in succession.

Having these principles and observations before us, let us resume the computation of the Tropical-Year-Train.

The number of hours'contained in the tropical year being 8765.813 208, the limits which we have prescribed to ourselves are 8765.813 196 and 8765.813 220. Instead, however, of at once restricting ourselves to this limit, we may seek for those trains which approach closely to the true ratio, and, having found a new train, draw closer and closer our limits as we proceed. The true ratio gives the quotients 8765, 1, 4, 2, 1, 4, 1, 4, 1, 7, 30, whence we obtain the following approximate values with their factors,

$$\frac{8765}{1} = \frac{5 \cdot 1753}{1}$$

$$\frac{8766}{1} = \frac{2 \cdot 3 \cdot 3 \cdot 487}{1}$$

$$\frac{43829}{5} = \frac{41 \cdot 1069}{5}$$

$$\frac{96424}{11} = \frac{2 \cdot 2 \cdot 2 \cdot 17 \cdot 709}{11}$$

$$\frac{140\,253}{16} = \frac{3 \cdot 46751}{2 \cdot 2 \cdot 2 \cdot 2}$$

$$\frac{657\,436}{75} = \frac{2 \cdot 2 \cdot 13 \cdot 47 \cdot 269}{3 \cdot 5 \cdot 5} = 8765^{h} \,..\, 48^{m} \,..\, 48^{s}.$$

$$\frac{797\,689}{91} = \frac{797\,689}{7 \cdot 13}$$

$$\frac{3\,848\,192}{439} = \frac{4 \cdot 8 \cdot 8 \cdot 8 \cdot 1879}{439}$$

$$\frac{4\,645\,881}{530} = \frac{3 \cdot 3 \cdot 516209}{2 \cdot 5 \cdot 53}$$

$$\frac{36\,369\,359}{4149} = \frac{13 \cdot 61 \cdot 45863}{3 \cdot 3 \cdot 461}$$

$$\frac{1\,095\,726651}{125\,000} = \frac{3 \cdot 31 \cdot 47 \cdot 250681}{2 \cdot 2 \cdot 2 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5}$$

The sixth of these is the only one of which the factors are sufficiently small; from it we can obtain a great variety of trains, but the error is considerable, amounting to $0^{s}.4536$, or nearly half a second in excess. Let us then seek for another train which shall give a result still nearer to the truth, or, allowing as much margin in defect, which shall give the length of the year between the limits

$$l = 8765.813\,082\,666$$

$$\mathrm{L} = 8765.813\,333\,333$$

The quotients for the first of these are 8765, 1, 4, 2, 1, 6, etc., and for the second 8765, 1, 4, 2, 1, 4, hence the two last coincident fractions are

$$\frac{p}{q} = \frac{96424}{11}; \quad \frac{r}{s} = \frac{140253}{16}$$

whence the limits to the value of the fraction $\frac{m}{n}$ are 4 and 6.01659. Hence all our subsequent fractions must be of the general form

$$\frac{140\,253\,m + 964\,24\,n}{16\,m + 11\,n}$$

the value of $m \div n$ being restricted to the above limits.

Putting first $n = 1$, we have $m = 5$ or 6, whence

$$\frac{m}{n} = \frac{5}{1}; \quad \frac{797\,689}{91} = \frac{797\,689}{7\,.\,13}$$

$$\frac{m}{n} = \frac{6}{1}; \quad \frac{937\,942}{107} = \frac{2\,.\,181\,.\,2591}{107}$$

neither of which gives a practical result: we therefore have recourse to fractions whose denominator is 2, and whose numerators are between 8 and 12.03318: hence

$$\frac{m}{n} = \frac{9}{2}; \quad \frac{1\,455\,125}{166} = \frac{5\,.\,5\,.\,5\,.\,7\,.\,1663}{2\,.\,83}$$

$$\frac{m}{n} = \frac{11}{2}; \quad \frac{1\,735\,631}{198} = \frac{19\,.\,167\,.\,547}{2\,.\,3\,.\,3\,.\,11} = 8765^{\mathrm{h}}\,..\,48^{\mathrm{m}}\,..\,47^{\mathrm{s}}.27$$

the latter of these is a close approximation, but 547 is rather large for the number of teeth in a wheel; it is only to be used in case that no better can be found; therefore, without restricting our limits, we shall try $n = 3$, with m between 12 and 18.04977; here

$$\frac{m}{n} = \frac{13}{3}; \quad \frac{2\,112\,561}{241} = \frac{3.3.3.11.2371}{241}$$

$$\frac{m}{n} = \frac{14}{3}; \quad \frac{2\,252\,814}{257} = \frac{2.3.239.1571}{257}$$

$$\frac{m}{n} = \frac{16}{3}; \quad \frac{2\,533\,320}{289} = \frac{2.2.2.3.3.5.31.227}{17.17} = 8765^{h}\,..\,48^{m}\,..\,47^{s}.335$$

$$\frac{m}{n} = \frac{17}{3}; \quad \frac{2\,673\,573}{305} = \frac{3.7.17.7489}{5.61}$$

The result just obtained, besides being a very close approximation, has the advantage of giving a *day wheel;* leaving out 2.2.2.3, the factors of 24, we have

$$\frac{3.5.31.227}{17.17} = 365\,\frac{70}{289} = 365.242\,214\,532$$

so that in 289 years there ought to be 70 intercalary days: Had the calendar been regulated in this way, the error in each cycle of 289 years would have been $0^{d}.000\,713$, or rather more than one minute in excess, and it would have required the lapse of 1400 of those cycles to have produced an error of one day.

Having now obtained a much better approximation, we must draw our limits still closer together, since we can derive no benefit from the discovery of a train which, though giving a result within the limits hitherto used, is more erroneous than the above train. We therefore make

$$l = 8765.813\,148\,789, \quad L = 8765.813\,267\,211:$$

with these we find the same coincident fractions, but the limits for $m \div n$ are 4.4040 and 5.3333.

Proceeding to fractions whose denominator is 4, we find

$$\frac{m}{n} = \frac{19}{4}; \quad \frac{3\,050\,503}{348} = \frac{101.30203}{2.2.3.29}$$

$$\frac{m}{n} = \frac{21}{4}; \quad \frac{3\,331\,009}{380} = \frac{11.11.27529}{2.2.5.19}$$

neither of which is of any use. Let us then have recourse to the denominator 5, omitting $\frac{24}{5}$, as it would only reproduce one of the original series;

$$\frac{m}{n}=\frac{22}{5}; \quad \frac{3\,567\,686}{407}=\frac{2\,.\,178\,3843}{11\,.\,37}$$

$$\frac{m}{n}=\frac{23}{5}; \quad \frac{3\,707\,939}{423}=\frac{\text{none under } 313}{3\,.\,3\,.\,47}$$

$$\frac{m}{n}=\frac{26}{5}; \quad \frac{4\,128\,698}{471}=\frac{2\,.\,7\,.\,79\,.\,3733}{3\,.\,157};$$

having again failed, let us try the denominator 6, omitting $\frac{29}{6}$, as giving one of the first series

$$\frac{m}{n}=\frac{31}{6}; \quad \frac{4\,926\,387}{562}=\frac{3\,.\,103\,.\,107\,.\,149}{2\,.\,281}=8765^h\,..\,48^m\,..\,47^s.402.$$

The error by this train is only the seventh part of a second, and, but for the inconvenience of having such a large number as 281 among the pinions, we might have been enabled still farther to contract our limits.

$$\frac{m}{n}=\frac{31}{7}; \quad \frac{5\,022\,811}{573}=\frac{\text{none under } 313}{3\,.\,191}$$

$$\frac{m}{n}=\frac{32}{7}; \quad \frac{5\,163\,064}{589}=\frac{2\,.\,2\,.\,2\,.\,645385}{19\,.\,31}$$

$$\frac{m}{n}=\frac{33}{7}; \quad \frac{5\,303\,317}{605}=\frac{23\,.\,29\,.\,7951}{5\,.\,11\,.\,11}$$

$$\frac{m}{n}=\frac{34}{7}; \quad \frac{5\,443\,570}{621}=\frac{2\,.\,5\,.\,11\,.\,17\,.\,41\,.\,71}{3\,.\,3\,.\,3\,.\,23}=8765^h\,..\,48^m\,..\,47^s.5362$$

$$\frac{m}{n}=\frac{36}{7}; \quad \frac{5\,724\,076}{653}=\frac{2\,.\,2\,.\,73\,.\,19603}{653}$$

$$\frac{m}{n}=\frac{37}{7}; \quad \frac{5\,864\,329}{662}=\frac{\quad}{3\,.\,223}$$

This solution enables us to represent the mean tropical year by help of very low-numbered wheels, and with almost all the precision of the present astronomical determinations. If we push the decimals of Bessel's result a little farther, we have $365^d.242\,217\,456$ days, or $8765^h.813\,218\,94$ hours for the tropical year: the above result is 8765.813 204 509, wherefore we may use now the new limits

$$l = 8765.813\,204\,509$$

$$L = 8765.813\,233\,379$$

which give the quotients

$$8765,\ 1,\ 4,\ 2,\ 1,\ 4,\ 1,\ \frac{7\,496\,575}{4\,237\,489} = 1.7691$$

$$8765,\ 1,\ 4,\ 2,\ 1,\ 4,\ 1,\ \frac{9\,661\,825}{1\,610\,319} = 5.9861$$

wherefore we must now operate with the fractions

$$\frac{p}{q} = \frac{797\,689}{91},\ \frac{r}{s} = \frac{657\,436}{75},$$

using m and n such that their quotient may be within the above limits.

In this way we may continue the search until we find some practicable train that may give a result still closer to the truth. With such high numbers the labour becomes considerable, as we can derive little help from the table of divisors. Much unnecessary toil may be avoided by treating the denominator first, since if it contain a large prime number, the train is impracticable.

When, from the nature of the case, we are required to obtain *absolutely* a given ratio, and when that ratio contains very high prime numbers, the simple train entirely fails us, and we must have recourse to some other expedient.

Let the proposed ratio be P : Q, in which P and Q involve large prime numbers. Assume some number R, susceptible of decomposition into moderate factors, then we may regard the ratio of P to Q as compounded of P : R and R : Q; and if we can obtain the one ratio by one piece of machinery, and the other by another, the combination of the two will give the ratio required. Again, if P have any moderate factors, these may, for the present, be left out, and the gist of our problem becomes this; to obtain the ratio P : R when P is a large indecomposable number, and when R is the continued product of moderate factors. Now, we can always find two decomposable numbers, S and T, whose sum or whose difference is P, so that each of the two ratios, S : R and T : R, may be obtainable from a simple train. If, then, we can devise any method of adding or subtracting the velocities given by two separate trains, we shall be able to obtain the desired ratio.

This addition or subtraction can be performed by the well-known contrivance called the *travelling* or *planet wheel.* In order to trace the nature of its action, let A, figure 68, be a pulley placed between two sliding bars, B and C, which are moving with the velocities s and t, then it is clear that the velocity of the centre of the pulley A is the arithmetical mean between s and t, being half their sum when B and C move in the same, and half their difference when B and C move in opposite directions. If B and C be bent into concentric circles, the same kind of action may be obtained: thus let the planet wheel A, figure 69, be placed in contact with the concave circumference B and with the convex circumference C of two wheels turning independently upon the common centre O: then it is obvious that the centre of A must describe a circle round O.

Let b and c be the radii of the two wheels, and, of course, $\frac{1}{2}(b-c)$ the radius of the planet wheel, then, if the angular velocities were for a moment arranged so that the centre of A may remain at rest, β', γ', α', being those temporary angular velocities, we must have

$$\beta' b = \tfrac{1}{2}(b - c)\,\alpha' = -\gamma' c\,;$$

and, if we now suppose the whole apparatus to be turned on the centre O with the angular velocity δ, we shall have δ for the velocity of angular transference of A. Put β and γ for the new angular velocities of the wheels, then $\beta = \beta' + \delta$; $\gamma = \gamma' + \delta$, whence $b\beta + c\gamma = (b + c)\delta$; now $b\beta$ and $c\gamma$ are the linear velocities of the two circumferences, and $\frac{1}{2}(b + c)\delta$ that of the centre A, and therefore, as before, the linear velocity of the centre of the planet wheel is the arithmetical mean of the velocities of the two circumferences. Hence

$$\delta = \frac{b\beta + c\gamma}{b + c}, \quad . \quad . \quad (95).$$

where b and c may be taken for the numbers of the teeth in the two wheels, and $\frac{1}{2}(b - c)$ for the number in the planet when they are all in one plane. But if B and C be placed in different planes, while A is made a bevelled wheel running between them, B and C may even be made equal to each other, and then the formula becomes simply $\delta = \frac{1}{2}(\beta + \gamma)$. In all cases the number of teeth in A is a matter of indifference.

Let us apply this arrangement to the year train. Representing the angular velocity of the year wheel by 1, that of the hour wheel must be

$$\frac{3 \,.\, 31 \,.\, 47 \,.\, 250681}{125000}\,;$$

now the numbers 3, 31, 47, 125000 can all be obtained in the usual way, so that it remains for us to consider the number 250681.

If we use the bevel-wheel arrangement, making $b = c$, we may neglect their special values: but otherwise we must previously determine what they are to be. Since we have to satisfy the equation

$$b\beta + c\gamma = 250\,681,$$

and since this number has no small divisor (it is in fact prime), b and c must be prime to each other; and again, since $b + c$ occurs as a divisor, we may simplify our train by taking it a divisor of 125000. Let us then assume $b = 149$, $c = 101$, whence $b + c = 250$; and we must determine β and γ to satisfy the indeterminate equation,

$$149\,\beta + 101\,\gamma = 250\,681.$$

Following the simple and efficacious plan of M. Legendre, we at once find

$$149 . 61 + 101 . 8 + 101 . 149 . 16 = 250\,681,$$

whence, confining our attention to those solutions in which β and γ are both positive, there result the seventeen following varieties:

$\beta = 61 = 61$	$\gamma = 2392 = 8 . 13 . 23$
$162 = 2 . 3 . 3 . 3 . 3$	$2243 = 2243$
$263 = 263$	$2094 = 2 . 3 . 349$
$364 = 2 . 2 . 7 . 13$	$1945 = 5 . 389$
$465 = 3 . 5 . 31$	$1796 = 2 . 2 . 449$
$566 = 2 . 283$	$1647 = 3 . 3 . 3 . 61$
$667 = 23 . 29$	$1498 = 2 . 7 . 107$
$768 = 4 . 4 . 4 . 4 . 3$	$1349 = 19 . 71$
$869 = 11 . 79$	$1200 = 4 . 4 . 3 . 5 . 5$
$970 = 2 . 5 . 97$	$1051 = 1051$
$1071 = 3 . 3 . 7 . 17$	$902 = 2 . 11 . 41$
$1172 = 2 . 2 . 293$	$753 = 3 . 251$
$1273 = 19 . 67$	$604 = 2 . 2 . 151$
$1374 = 2 . 3 . 229$	$455 = 5 . 7 . 13$
$1475 = 5 . 5 . 59$	$306 = 2 . 3 . 3 . 17$
$1576 = 2 . 2 . 2 . 197$	$157 = 157$
$1677 = 3 . 13 . 43$	$8 = 2 . 2 . 2$

Here we have several practicable arrangements: guiding our selection by the advantage of having the angular velocities nearly equal to each other, we may take

$$11 . 79 . 149 + 16 . 75 . 101 = 250681.$$

We must connect the axis of the outer wheel B of 149 teeth with the year-shaft, so as to cause it to make $11.79 = 869$ turns per annum; at the same time the inner wheel C of 101 teeth must be made to turn $16.75 = 1200$ times. With this arrangement the arm carrying the planet wheel will make $250681 \div 250$ turns in a year; so that we must still supply the ratio $3.31.47 \div 500$, in order to obtain the required 8765.813 208 turns per annum: this can be done in a great variety of ways.

In this class of arrangements there are three parts turning, with different angular velocities, on one axis: to allow of this two of the supports must be tubular, or the axes must be supported from separate plates.

A still greater range of selection may be obtained by placing, instead of the simple wheel A, two wheels of different diameters fixed together, the one of them acting on B, and the other on C. This arrangement is represented in figure 70.

If, as before, b and c be the numbers of teeth in the wheels B and C, and if A be the number in that one of the wheels A which gears into B, a in that one which gears into C, it is easy to shew that

$$\delta = \frac{a b \beta + A c \gamma}{a b + A c} \quad . \quad . \quad (96).$$

Here it is not necessary, as it was in the preceding case, that the teeth of B and c have the same pitch: the contact b, A, may have one, and the contact c, a, another size of tooth. This arrangement allows us immense scope in the selection of our numbers, so that we are enabled at the same time to attend to correlated matters. The wheels may be bevelled without altering the above equation.

Even still greater complexity may be given to the travelling part: thus on the arm which supported A we may place, instead of the one wheel, a whole train of wheels, the one extremity of which gears into B, and the other extremity into C; in this arrangement we need not employ internal wheels.

Let, as before, B and C, now supposed both to be convex, have b and c teeth, B being geared into a wheel of a teeth, and C into a wheel of e teeth, whose revolutions in respect to their own system are arranged so that their angular velocities are in the ratio $\alpha : \epsilon$. Then, supposing the arm which supports the travelling train to be stationary, it is easy to shew that

$$\alpha a = - \beta' b ; \; \epsilon e = - \gamma' c ; \text{ whence}$$

$$\delta = \frac{b \beta . e \epsilon - c \gamma . a \alpha}{b e \epsilon - c a \alpha} \quad . \quad . \quad (97).$$

which allows us any desired range of selection.

In this way there is no imaginable degree of precision which cannot be attained by help of wheel-work, yet, since in all physical determinations we can only pretend to a moderate degree of exactitude, it is at all times expedient to try for a simple train, and to have recourse to the travelling arm only when the other fails us.

The student whose attention is, for the first time, directed to this subject, will find advantage in making some of the computations for the mean motions of the planets, as if he were about to construct an orrery: such practice will remove any difficulty or obscurity which he may have encountered in perusing the preceding pages, and will also open up interesting views of the highly important but much neglected Theory of Numbers. It will also afford him excellent exercise to compute the relative pressures on the teeth of trains arranged as with the planet-wheel.

APPENDIX.

TABLE I.

Hour-Glass Curves.

n	X	Y	i	ω	QM	QN	QO	k
1	0.5000	1.0000	45°··00′	90°··00′	1.000	0.000	0.000	1.00
3	2.3017	2.1455	24··59	36··41	7.731	3.610	4.502	.47
5	3.8949	2.7910	19··43	28··29	18.950	7.092	10.779	.36
7	5.4749	3.3091	16··49	24··09	32.925	11.243	18.530	.30
9	7.0508	3.7552	14··55	21··20	49.200	15.972	27.523	.27
11	8.6249	4.1533	13··32	19··20	67.490	21.219	37.608	.24
13	10.1979	4.5162	12··29	17··48	87.595	26.936	48.678	.22
15	11.7703	4.8519	11··39	16··35	109.365	33.087	60.652	.21
17	13.3424	5.1657	10··57	15··35	132.681	39.646	73.469	.19
19	14.9142	5.4615	10··22	14··45	157.447	46.586	87.075	.18
21	16.4858	5.7421	9··53	14··03	183.583	53.890	101.426	.17
23	18.0572	6.0095	9··27	13··25	211.021	61.539	116.489	.17
NUMBER OF POINTS IN CONTACT.	HALF BREADTH OF HOUR-GLASS CURVE.	HALF LENGTH OF HOUR-GLASS CURVE.	LEAST OBLIQUITY OF ACTION.	GREATEST OBLIQUITY OF ACTION.	LIMIT OF COMPLETE MECHANICAL CONTACT.	LIMIT OF COMPLETE ACTION AT THE SHOULDER.	LIMIT OF COMPLETE PRAGMATIC ACTION.	RADIUS OF CURVATURE AT BOTTOM OF RACK.

TABLE II. TABLE III.

	One Contact.		Three Contacts.			
	Tracing Point.		1st Point.		2d Point.	
a	x	y	x	y	x	y
0°	0.0000	0.0000	± 2.2467	∓ 1.6738	0.0000	0.0000
9	+ 0.0784	+ 0.0786	− 2.2751	+ 1.3975	+ 0.1290	+ 0.0602
18	0.1558	0.1577	2.1949	1.2684	0.2580	0.1204
27	0.2311	0.2379	2.0973	1.1634	0.3868	0.1809
36	0.3029	0.3197	1.9908	1.0707	0.5154	0.2418
45	0.3695	0.4040	1.8787	0.9854	0.6347	0.3030
54	0.4283	0.4919	1.7629	0.9065	0·7716	0.3649
63	0.4745	0.5851	1.6442	0.8312	0.8989	0.4276
72	0.4992	0.6870	1.5233	0.7591	1.0257	0.4911
81	+ 0.4774	0.8054	1.4007	0.6894	1.1517	0.5558
90	0.0000	1.0000	1.2768	0.6218	1.2768	0.6218
99	− 0.4774	0.8054	1.1517	0.5558	1.4007	0.6894
108	0.4992	0.6870	1.0257	0.4911	1.5233	0.7591
117	0.4745	0.5851	0.8989	0.4276	1.6442	0.8313
126	0.4283	0.4919	0.7716	0.3649	1.7629	0.9065
135	0.3695	0.4040	0.6347	0.3030	1.8787	0.9854
144	0.3029	0.3197	0.5154	0.2418	1.9908	1.0707
153	0.2311	0.2379	0.3868	0.1809	2.0973	1.1634
162	0.1558	0.1577	0.2580	0.1204	2.1949	1.2684
171	− 0.0784	+ 0.0786	− 0.1290	+ 0.0602	+ 2.2751	+ 1 3975
180	0.0000	0.0000	0.0000	0.0000	± 2.2467	± 1.6738
189	+ 0.0784	− 0.0786	+ 0.1290	− 0.0602	− 2.2751	− 1.3975
198	0.1558	0.1577	0.2580	0.1204	2.1949	1.2684
207	0.2311	0.2379	0.3868	0.1809	2.0973	1.1634
216	0.3029	0.3197	0.5154	0.2418	1.9908	1.0707
225	0.3695	0.4040	0.6347	0.3030	1.8787	0.9854
234	0.4283	0.4919	0.7716	0.3649	1.7629	0.9065
243	0.4745	0.5851	0.8989	0.4276	1.6442	0.8313
252	0.4992	0.6870	1.0257	0.4911	1.5233	0.7591
261	+ 0.4774	0.8054	1.1517	0.5558	1.4007	0.6894
270	0.0000	1.0000	1.2768	0.6218	1.2768	0.6218
279	− 0.4774	0.8054	1.4007	0.6894	1.1517	0.5558
288	0.4992	0.6870	1.5233	0.7591	1.0257	0.4911
297	0.4745	0.5851	1.6442	0.8313	0.8989	0.4276
306	0.4283	0.4919	1.7629	0.9065	0.7716	0.3649
315	0.3695	0.4040	1.8787	0.9854	0.6347	0.3030
324	0.3029	0 3197	1.9908	1.0707	0.5154	0.2418
333	0.2311	0.2379	2.0973	1.1634	0.3868	0.1809
342	0.1558	0.1577	2.1949	1.2684	0.2580	0.1204
351	− 0.0784	− 0.0786	+ 2.2751	− 1.3975	− 0.1290	− 0.0602
360	0.0000	0.0000	± 2.2467	∓ 1.6738	0.0000	0.0000

	Three Contacts.		**Five Contacts.**			
	3d Point.		**1st Point.**		**2d Point.**	
α	x	y	x	y	x	y
0°	± 2.2467	∓ 1.6738	− 2.7497	+ 1.0660	0.0000	0.0000
9	− 1.8988	+ 1.8980	2.6162	1.0048	+ 0.1392	+ 0.0499
18	1.6951	1.9741	2.4820	0.9451	0.2784	0.0998
27	1.4366	2.0248	2.3470	0.8869	0.4175	0.1498
36	1.2224	2.0620	2.2114	0.8299	0.5556	0.1999
45	1.0133	2.0900	2.0752	0.7739	0.6956	0.2502
54	0.8076	2.1112	1.9386	0.7188	0.8345	0.3008
63	0.6040	2.1267	1.8016	0.6646	0.9733	0.3515
72	0.4019	2.1373	1.6642	0.6111	1.1119	0.4026
81	− 0.2007	2.1435	1.5265	0.5582	1.2503	0.4540
90	0.0000	2.1455	1.3885	0.5059	1.3885	0.5059
99	+ 0.2007	2.1435	1.2503	0.4540	1.5265	0.5582
108	0.4019	2.1373	1.1119	0.4026	1.6642	0.6111
117	0.6040	2.1267	0.9733	0.3515	1.8016	0.6646
126	0.8076	2.1112	0.8345	0.3008	1.9386	0.7188
135	1.0133	2.0900	0.6956	0.2502	2.0752	0.7739
144	1.2224	2.0620	0.5566	0.1999	2.2114	0.8299
153	1.4366	2.0248	0.4175	0.1498	2.3470	0.8869
162	1.6591	1.9741	0.2784	0.0998	2.4820	0.9451
171	+ 1.8988	+ 1.8980	− 0.1392	+ 0.0499	2.6162	1.0048
180	± 2.2467	± 1.6738	0.0000	0.0000	2.7497	1.0660
189	− 1.8988	− 1.8980	+ 0.1392	− 0.0499	2.8821	1.1292
198	1.6591	1.9741	0.2784	0.0998	3.0134	1.1947
207	1.4366	2.0248	0.4175	0.1498	3.1433	1.2629
216	1.2224	2.0620	0.5566	0.1999	3.2713	1.3346
225	1.0133	2.0900	0.6956	0.2502	3.3971	1.4106
234	0.8076	2.1112	0.8345	0.3008	3.5199	1.4925
243	0.6040	2.1267	0.9733	0.3515	3.6383	1.5826
252	0.4019	2.1373	1.1119	0.4026	3.7497	1.6857
261	− 0.2007	2.1435	1.2503	0.4540	+ 3.8477	1.8141
270	0.0000	2.1455	1.3885	0.5059	± 3.8626	2.0964
279	+ 0.2007	2.1435	1.5265	0.5582	− 3.8477	1.8141
288	0.4019	2.1373	1.6642	0.6111	3.7497	1.6857
297	0.6040	2.1267	1.8016	0.6646	3.6383	1.5826
306	0.8076	2.1112	1.9386	0.7188	3.5199	1.4925
315	1.0133	2.0900	2.0752	0.7739	3.3971	1.4106
324	1.2224	2.0620	2.2114	0.8299	3.2713	1.3346
333	1.4366	2.0248	2.3470	0.8869	3.1433	1.2629
342	1.6591	1.9741	2.4820	0.9451	3.0134	1.1947
351	+ 1.8988	− 1.8980	2.6162	1.0048	2.8821	1.1292
360	± 2.2467	∓ 1.6738	+ 2 7497	− 1.0660	− 2.7947	+ 1.0660

TABLE IV.

Five Contacts.						
	3d Point.		4th Point.		5th Point.	
α	x	y	x	y	x	y
0°	+ 2.7497	− 1.0660	− 1.8129	+ 2.7096	+ 1.8129	− 2.7096
9	2.8821	1.1292	1.6296	2.7262	1.9971	2.6905
18	3.0134	1.1947	1.4470	2.7406	2.1823	2.6685
27	3.1433	1.2629	1.2649	2.7529	2.3689	2.6432
36	3.2713	1.3346	1.0834	2.7633	2.5572	2.6139
45	3.3971	1.4106	0.9023	2.7720	2.7477	2.5798
54	3.5199	1.4925	0.7215	2.7789	2.9412	2.5393
63	3.6383	1.5826	0.5409	2.7842	3.1390	2.4901
72	3.7497	1.6857	0.3605	2.7880	3.3436	2.4274
81	+ 3.8477	1.8141	− 0.1802	2.7903	+ 3.5616	2.3391
90	± 3.8626	2.0964	0.0000	2.7910	± 3.8626	2.0964
99	− 3.8477	1.8141	+ 0.1802	2.7903	− 3.5616	2.3391
108	3.7497	1.6857	0.3605	2.7880	3.3436	2.4274
117	3.6383	1.5826	0.5409	2.7842	3.1390	2.4901
126	3.5199	1.4925	0.7215	2.7789	2.9412	2.5393
135	3.3971	1.4106	0.9023	2.7720	2.7477	2.5798
144	3.2713	1.3346	1.0834	2.7633	2.5572	2.6139
153	3.1433	1.2629	1.2649	2.7529	2.3689	2.6432
162	3.0134	1.1947	1.4470	2.7406	2.1823	2.6685
171	2.8821	1.1292	1·6296	2.7262	1.9971	2.6905
180	2.7497	1.0660	1.8129	2.7096	1.8129	2.7096
189	2.6162	1.0048	1.9971	2.6905	1.6296	2.7262
198	2.4820	0.9451	2.1823	2.6685	1.4470	2.7406
207	2.3470	0.8869	2.3689	2.6432	1.2649	2.7529
216	2.2114	0.8299	2.5572	2.6139	1.0834	2.7633
225	2.0752	0.7739	2.7477	2.5798	0.9023	2.7720
234	1.9386	0.7188	2.9412	2.5393	0.7215	2.7789
243	1.8016	0.6646	3.1390	2.4901	0.5409	2.7842
252	1.6642	0.6111	3.3436	2.4274	0.3605	2.7880
261	1.5265	0.5582	+ 3.5616	2.3391	− 0.1802	2.7903
270	1.3885	0.5059	± 3.8626	2.0964	0.0000	2.7910
279	1.2503	0.4540	− 3.5616	2.3391	+ 0.1802	2.7903
288	1.1119	0.4026	3.3436	2.4274	0.3605	2.7880
297	0.9733	0.3515	3.1390	2.4901	0.5409	2.7842
306	0.8345	0.3008	2.9412	2.5393	0.7215	2.7789
315	0.6956	0.2502	2.7477	2.5798	0.9023	2.7720
324	0.5566	0.1999	2.5572	2.6139	1.0834	2.7633
333	0.4176	0.1498	2.3689	2.6432	1.2649	2.7529
342	0.2784	0.0998	2.1823	2.6685	1.4470	2.7406
351	− 0.1392	− 0.0499	1.9971	2.6905	1.6296	2.7262
360	0.0000	0.0000	− 1.8129	+ 2.7096	+ 1.8129	− 2.7096

TABLE V.

Seven Contacts.

	1st Point.		2d Point.		3d Point.	
a	x	y	x	y	x	y
0°	± 5.4521	± 2.4444	– 2.8661	+ 0·9001	0.0000	0.0000
9	– 5.4149	– 2.1601	2.7241	0.8520	+ 0.1439	+ 0.0435
18	5.3076	2.0326	2.5819	0.8044	0.2879	0.0870
27	5.1889	1.9309	2.4394	0.7573	0.4318	0.1306
36	5.0643	1.8425	2.2968	0.7107	0.5756	0.1742
45	4.9361	1.7625	2.1540	0.6645	0.7195	0.2719
54	4.8052	1.6885	2.0111	0.6186	0.8633	0.2617
63	4.6724	1.6190	1.8679	0.5732	1.0071	0.3056
72	4.5380	1.5531	1.7247	0.5280	1.1508	0.3497
81	4.4024	1.4900	1.5814	0.4831	1.2944	0.3940
90	4.2657	1.4293	1.4379	0.4384	1.4379	0.4384
99	4.1282	1.3707	1.2944	0.3940	1.5814	0.4831
108	3.9900	1.3137	1.1508	0.3497	1.7247	0.5280
117	3.8511	1.2583	1.0071	0.3056	1.8679	0.5732
126	3.7116	1.2042	0.8633	0.2617	2.0111	0.6186
135	3.5716	1.1513	0.7195	0.2179	2.1540	0.6645
144	3.4312	1.0994	0.5756	0.1742	2.2968	0.7107
153	3.2905	1.0484	0·4318	0.1306	2.4394	0.7573
162	3.1493	0.9982	0.2879	0.0870	2.5819	0.8044
171	3.0079	0.9488	– 0.1439	+ 0.0435	2.7241	0.8520
180	2.8661	0.9001	0.0000	0.0000	2.8661	0.9001
189	2.7241	0.8520	+ 0.1439	– 0.0435	3.0079	0.9488
198	2.5819	0.8044	0.2879	0.0870	3.1493	0.9982
207	2.4394	0.7573	0.4318	0.1306	3.2905	1.0484
216	2.2968	0.7107	0.5756	0.1742	3.4312	1.0994
225	2.1540	0.6645	0.7195	0.2179	3.5716	1.1513
234	2.0111	0.6186	0.8633	0.2617	3.7116	1.2042
243	1.8769	0.5732	1.0071	0.3056	3.8511	1.2583
252	1.7247	0.5280	1.1508	0.3497	3.9900	1.3137
261	1.5814	0.4831	1.2944	0.3940	4.1282	1.3707
270	1.4379	0.4384	1.4379	0.4384	4.2657	1.4293
279	1.2944	0.3940	1.5814	0.4831	4.4024	1.4900
288	1.1508	0.3497	1.7247	0.5280	4.5380	1.5531
297	1.0071	0.3056	1.8679	0.5732	4.6724	1.6190
306	0.8633	0.2617	2.0111	0.6186	4.8052	1.6885
315	0.7195	0.2179	2.1540	0.6645	4.9361	1.7625
324	0.5756	0.1742	2.2968	0.7107	5.0643	1.8425
333	0.4318	0.1306	2.4394	0.7573	5.1889	1.9309
342	0.2879	0.0870	2.5819	0.8044	5.3076	2.0326
351	– 0.1439	– 0.0435	2.7241	0.8520	+ 5.4149	+ 2.1601
360	0.0000	0.0000	+ 2.8661	– 0.9001	± 5.4521	± 2.4444

TABLE V.

Seven Contacts.						
	4th Point.		5th Point.		6th Point.	
α	x	y	x	y	x	y
0°	+ 2.8661	− 0.9001	± 5.4521	± 2.4444	− 1.7317	+ 3.2663
9	3.0079	0.9488	− 5.1742	− 2.6953	1.5580	3.2747
18	3.1493	0.9982	4.9664	2.7891	1.3845	3.2821
27	3.2905	1.0484	4.7700	2.8570	1.2111	3.2885
36	3.4312	1.0994	4.5794	2.9113	1.0378	3.2940
45	3.5716	1.1513	4.3926	2.9569	0.8470	3.2987
54	3.7116	1.2042	4.2083	2.9962	0.6917	3.3024
63	3.8511	1.2583	4.0259	3.0307	0.5187	3.3053
72	3.9900	1.3137	3.8451	3.0614	0.3458	3.3074
81	4.1282	1.3707	3.6654	3.0890	− 0.1729	3.3086
90	4.2657	1.4293	3.4868	3.1138	0.0000	3.3091
99	4.4024	1.4900	3.3090	3.1364	+ 0.1729	3.3086
108	4.5380	1.5531	3.1319	3.1568	0.3458	3.3074
117	4.6724	1.6190	2.9555	3.1755	0.5187	3.3053
126	4.8052	1.6885	2.7795	3.1925	0.6917	3.3024
135	4.9361	1.7625	2.6040	3.2080	0.8647	3.2987
144	5.0643	1.8425	2.4289	3.2220	1.0378	3.2940
153	5.1889	1.9309	2.2542	3.2349	1.2111	3.2885
162	5.3076	2.0326	2.0798	3.2465	1.3845	3.2821
171	+ 5.4149	− 2.1601	1.9056	3.2569	1.5580	3.2747
180	± 5.4521	∓ 2.4444	1.7317	3.2663	1.7317	3.2663
189	− 5.4149	+ 2.1601	1.5580	3.2747	1.9056	3.2569
198	5.3076	2.0326	1.3845	3.2821	2.0798	3.2465
207	5.1889	1.9309	1.2111	3.2885	2.2542	3.2349
216	5.0643	1.8425	1.0378	3.2940	2.4289	3.2220
225	4.9361	1.7625	0.8647	3.2987	2.6040	3.2080
234	4.8052	1.6885	0.6917	3.3024	2.7795	3.1925
243	4.6724	1.6190	0.5187	3.3053	2.9555	3.1755
252	4.5380	1.5531	0.3458	3.3074	3.1319	3.1568
261	4.4024	1.4900	− 0.1729	3.3086	3.3090	3.1364
270	4.2657	1.4293	0.0000	3.3091	3.4868	3.1138
279	4.1282	1.3707	+ 0.1729	3.3086	3.6654	3.0890
288	3.9900	1.3137	0.3458	3.3074	3.8451	3.0614
297	3.8511	1.2583	0.5187	3·3053	4.0259	3.0307
306	3.7116	1.2042	0.6917	3.3024	4.2083	2.9962
315	3.5716	1.1513	0.8647	3.2987	4.3926	2.9569
324	3.4312	1.0994	1.0378	3.2940	4.5794	2.9113
333	3.2905	1.0484	1.2111	3.2885	4.7700	2.8570
342	3.1493	0.9982	1.3845	3.2821	4.9664	2.7891
351	3.0079	0.9488	1.5580	3.2747	+ 5.1742	+ 2.6953
360	− 2.8661	+ 0.9001	+ 1 7317	− 3.2663	± 5.4521	± 2.4444

	Seven Contacts.		Nine Contacts.			
	7th Point.		1st Point.		2d Point.	
α	*x*	*y*	*x*	*y*	*x*	*y*
0°	+ 1.7317	− 3.2663	− 5.8002	− 1.7441	− 2.9275	+ 0.7978
9	1.9056	3.2569	5.6601	1.6871	2.7817	0.7563
18	2.0798	3.2465	5.5194	1.6320	2.6359	0.7150
27	2.2542	3.2349	5.3782	1.5784	2.4899	0.6740
36	2.4289	3.2220	5.2364	1.5261	2.3438	0.6332
45	2.6040	3.2080	5.0941	1.4751	2.1977	0.5927
54	2.7795	3.1925	4.9514	1.4252	2.0515	0.5523
63	2.9555	3.1755	4.8084	1.3762	1.9052	0.5121
72	3.1319	3.1568	4.6650	1.3281	1.7589	0.4721
81	3.3090	3.1364	4.5214	1.2808	1.6125	0.4323
90	3.4868	3.1138	4.3775	1.2343	1.4661	0.3926
99	3.6654	3.0890	4.2333	1.1884	1.3196	0.3530
108	3.8451	3.0614	4.0889	1.1431	1.1731	0.3135
117	4.0259	3.0307	3.9443	1.0984	1.0265	0.2741
126	4.2083	2.9962	3.7995	1.0542	0.8799	0.2348
135	4.3926	2.9569	3.6546	1.0105	0.7333	0.1955
144	4.5794	2.9113	3.5094	0.9672	0.5867	0.1564
153	4.7700	2.8570	3.3641	0.9243	0.4400	0.1172
162	4.9664	2.7891	3.2187	0.8818	0.2934	0.0781
171	+ 5.1742	− 2.6953	3.0732	0.8396	− 0.1467	+ 0.0391
180	± 5.4521	∓ 2.4444	2.9275	0.7978	0.0000	0.0000
189	− 5.1742	+ 2.6953	2.7817	0.7563	+ 0.1467	− 0.0391
198	4.9664	2.7891	2.6359	0.7150	0.2934	0.0781
207	4.7700	2.8570	2.4899	0.6740	0.4400	0.1172
216	4.5794	2.9113	2.3438	0.6332	0.5867	0.1564
225	4.3926	2.9569	2.1977	0.5927	0.7333	0.1955
234	4.2083	2.9962	2.0515	0.5523	0.8799	0.2348
243	4.0259	3.0307	1.9052	0.5121	1.0265	0.2741
252	3.8451	3.0614	1.7589	0.4721	1.1731	0.3135
261	3.6654	3.0890	1.6125	0.4323	1.3196	0.3530
270	3.4868	3.1138	1.4661	0.3926	1.4661	0.3926
279	3.3090	3.1364	1.3196	0.3530	1.6125	0.4323
288	3.1319	3.1568	1.1731	0.3135	1.7589	0.4721
297	2.9555	3.1755	1.0265	0.2741	1.9052	0.5121
306	2.7795	3.1925	0.8799	0.2348	2.0515	0.5523
315	2.6040	3.2080	0.7333	0.1955	2.1977	0.5927
324	2.4289	3.2220	0.5867	0.1564	2.3438	0.6332
333	2.2542	3.2349	0.4400	0.1172	2.4899	0.6740
342	2.0798	3.2465	0.2934	0.0781	2.6359	0.7150
351	1.9056	3.2569	− 0.1467	− 0.0391	2.7817	0.7563
360	− 1.7317	+ 3.2663	0.0000	0.0000	+ 2.9275	− 0.7978

TABLE VI.

NINE CONTACTS.						
	3d Point.		4th Point.		5th Point.	
a	*x*	*y*	*x*	*y*	*x*	*y*
0°	0.0000	0.0000	+ 2.9275	− 0.7978	+ 5.8002	+ 1.7441
9	+ 0.1467	+ 0.0391	3.0732	0.8396	5.9394	1.8031
18	0.2934	0.0781	3.2187	0.8818	6.0778	1.8646
27	0.4400	0.1172	3.3641	0.9243	6.2150	1.9290
36	0.5867	0.1564	3.5094	0.9672	6.3508	1.9970
45	0.7333	0.1955	3.6546	1.0105	6.4849	2.0696
54	0.8799	0.2348	3.7995	1.0542	6.6167	2.1482
63	1.0265	0.2741	3.9443	1.0984	6.7451	2.2354
72	1.1731	0.3135	4.0889	1.1431	6.8684	2.3361
81	1.3196	0.3530	4.2333	1.1884	+ 6.9816	2.4628
90	1.4661	0.3926	4.3775	1.2343	± 7.0331	2.7479
99	1.6125	0.4323	4.5214	1.2808	− 6.9816	2.4628
108	1.7589	0.4721	4.6651	1.3281	6.8684	2.3361
117	1.9052	0.5121	4.8084	1.3762	6.7451	2.2354
126	2.0515	0.5523	4.9514	1.4252	6.6167	2.1482
135	2.1977	0.5927	5.0941	1.4751	6.4849	2.0696
144	2.3438	0.6332	5.2364	1.5261	6.3508	1.9970
153	2.4899	0.6740	5.3782	1.5784	6.2150	1.9290
162	2.6359	0.7150	5.5194	1.6320	6.0778	1.8646
171	2.7817	0.7563	5.6601	1.6871	5.9394	1.8031
180	2.9275	0.7978	5.8002	1.7441	5.8002	1.7441
189	3.0732	0.8396	5.9394	1.8031	5.6601	1.6871
198	3.2187	0.8818	6.0778	1.8646	5.5194	1.6320
207	3.3641	0.9243	6.2150	1.9290	5.3782	1.5784
216	3.5094	0.9672	6.3508	1.9970	5.2364	1.5261
225	3.6546	1.0105	6.4849	2.0696	5.0941	1.4751
234	3.7995	1.0542	6.6167	2.1482	4.9514	1.4252
243	3.9443	1.0984	6.7451	2.2354	4.8084	1.3762
252	4.0889	1.1431	6.8684	2.3361	4.6651	1.3281
261	4.2333	1.1884	+ 6.9816	2.4628	4.5214	1.2808
270	4.3775	1.2343	± 7.0331	2.7479	4.3775	1.2343
279	4.5214	1.2808	− 6.9816	2.4628	4.2333	1.1884
288	4.6651	1.3281	6.8684	2.3361	4.0889	1.1431
297	4.8084	1.3762	6.7451	2.2354	3.9443	1.0984
306	4.9514	1.4252	6.6167	2.1482	3.7995	1.0542
315	5.0941	1.4751	6.4849	2.0696	3.6546	1.0105
324	5.2364	1.5261	6.3508	1.9970	3.5094	0.9672
333	5.3782	1.5784	6.2150	1.9290	3.3641	0.9243
342	5.5194	1.6320	6.0778	1.8646	3.2187	0.8818
351	5.6601	1.6871	5.9394	1.8031	3.0732	0.8396
360	+ 5.8002	+ 1.7441	− 5.8002	− 1.7441	− 2.9275	+ 0.7978

TABLE VI.

NINE CONTACTS.

	6th Point.		7th Point.		8th Point.	
a	x	y	x	y	x	y
0°	− 5.1186	− 3.4496	− 1.6920	+ 3.7277	+ 1.6920	− 3.7277
9	4.9438	3.4754	1.5225	3.7330	1.8615	3.7218
18	4.7697	3.4991	1.3532	3.7377	2.0311	3.7152
27	4.5962	3.5210	1.1839	3.7419	2.2008	3.7080
36	4.4231	3.5414	1.0147	3.7454	2.3706	3.7002
45	4.2505	3.5604	0.8455	3.7484	2.5405	3.6916
54	4.0783	3.5781	0.6764	3.7509	2.7106	3.6824
63	3.9064	3.5945	0.5072	3.7528	2.8808	3.6724
72	3.7349	3.6099	0.3381	3.7541	3.0512	3.6616
81	3.5636	3.6242	− 0.1691	3.7550	3.2218	3.6500
90	3.3926	3.6376	0.0000	3.7552	3.3926	3.6376
99	3.2218	3.6500	+ 0.1691	3.7550	3.5636	3.6242
108	3.0512	3.6616	0.3381	3.7541	3.7349	3.6099
117	2.8808	3.6724	0.5072	3.7528	3.9064	3.5945
126	2.7106	3.6824	0.6764	3.7509	4.0783	3.5781
135	2.5405	3.6916	0.8455	3.7484	4.2505	3.5604
144	2.3706	3.7002	1.0147	3.7454	4.4231	3.5414
153	2.2008	3.7080	1.1839	3.7419	4.5962	3.5210
162	2.0311	3.7152	1.3532	3.7377	4.7697	3.4991
171	1.8615	3.7218	1.5225	3.7330	4.9438	3.4754
180	1.6920	3.7277	1.6920	3.7277	5.1186	3.4496
189	1.5225	3.7330	1.8615	3.7218	5.2941	3.4217
198	1.3532	3.7377	2.0311	3.7152	5.4706	3.3910
207	1.1839	3.7419	2.2008	3.7080	5.6482	3.3573
216	1.0147	3.7454	2.3706	3.7002	5.8271	3.3197
225	0.8455	3.7484	2.5405	3.6916	6.0077	3.2774
234	0.6764	3.7509	2.7106	3.6824	6.1907	3.2289
243	0.5072	3.7528	2.8808	3.6724	6.3770	3.1716
252	0.3381	3.7541	3.0512	3.6616	6.5684	3.1007
261	− 0.1691	3.7550	3.2218	3.6500	+ 6.7699	3.0036
270	0.0000	3.7552	3.3926	3.6376	± 7.0331	3.7479
279	+ 0.1691	3.7550	3.5636	3.6242	− 6.7699	3.0036
288	0.3381	3.7541	3.7349	3.6099	6.5684	3.1007
297	0.5072	3.7528	3.9064	3.5945	6.3770	3.1716
306	0.6764	3.7509	4.0783	3.5781	6.1907	3.2289
315	0.8455	3.7484	4.2505	3.5604	6.0077	3.2774
324	1.0147	3.7454	4.4231	3.5414	5.8271	3.3197
333	1.1839	3.7419	4.5962	3.5210	5.6482	3.3573
342	1.3532	3.7377	4.7697	3.4991	5.4706	3.3910
351	1.5225	3.7330	4.9438	3.4754	5.2941	3.4217
360	+ 1.6920	− 3.7277	+ 5.1186	+ 3.4496	− 5.1186	− 3.4496

TABLE VI.

TABLE VII.

	Nine Contacts.		Eleven Contacts.			
	9th Point.		1st Point.		2d Point.	
a	x	y	x	y	x	y
0°	+ 5.1186	+ 3.4496	± 8.6104	∓ 3.0207	– 5.9060	– 1.5295
9	5.2941	3.4217	– 8.5488	+ 2.7353	5.7604	1.4852
18	5.4706	3.3910	8.4313	2.6092	5.6146	1.4414
27	5.6482	3.3573	8.3048	2.5094	5.4686	1.3983
36	5.8271	3.3198	8.1735	2.4231	5.3223	1.3557
45	6.0077	3.2774	8.0393	2.3454	5.1760	1.3137
54	6.1907	3.2289	7.9029	2.2739	5.0294	1.2721
63	6.3770	3.1716	7.7650	2.2070	4.8827	1.2309
72	6.5684	3.1007	7.6258	2.1437	4.7359	1.1902
81	+ 6.7699	3.0036	7.4856	2.0834	4.5890	1.1498
90	± 7.0331	2.7479	7.3445	2.0256	4.4419	1.1098
99	– 6.7699	3.0036	7.2027	1.9699	4.2947	1.0702
108	6.5684	3.1007	7.0603	1.9160	4.1474	1.0308
117	6.3770	3.1716	6.9174	1.8637	4.0000	0.9918
126	6.1907	3.2289	6.7740	1.8128	3.8525	0.9530
135	6.0077	3.2774	6.6302	1.7631	3.7050	0.9145
144	5.8271	3.3197	6.4860	1.7146	3.5573	0.8762
153	5.6482	3.3573	6.3414	1.6670	3.4096	0.8382
162	5.4706	3.3910	6.1966	1.6204	3.2618	0.8004
171	5.2941	3.4217	6.0514	1.5746	3.1140	0.7627
180	5.1186	3.4496	5.9060	1.5295	2.9661	0.7253
189	4.9438	3.4754	5.7604	1.4852	2.8181	0.6880
198	4.7697	3.4991	5.6146	1.4414	2.6701	0.6509
207	4.5962	3.5210	5.4686	1.3983	2.5220	0.6140
216	4.4231	3.5414	5.3223	1.3557	2.3739	0.5772
225	4.2505	3.5604	5.1760	1.3137	2.2257	0.5405
234	4.0783	3.5781	5.0294	1.2721	2.0775	0.5039
243	3.9064	3.5945	4.8827	1.2309	1.9292	0.4675
252	3.7349	3.6099	4.7359	1.1902	1.7810	0.4311
261	3.5636	3.6242	4.5890	1.1498	1.6327	0.3949
270	3.3926	3.6376	4.4419	1.1098	1.4843	0.3587
279	3.2218	3.6500	4.2947	1.0702	1.3360	0.3226
288	3.0512	3.6616	4.1474	1.0308	1.1876	0.2866
297	2.8808	3.6724	4.0000	0.9918	1.0392	0.2507
306	2.7106	3.6824	3.8525	0.9530	0.8907	0.2148
315	2.5405	3.6916	3.7050	0.9145	0.7423	0.1789
324	2.3706	3.7002	3.5573	0.8762	0.5939	0.1431
333	2.2008	3.7080	3.4096	0.8382	0.4454	0.1073
342	2.0311	3.7152	3.2618	0.8004	0.2969	0.0715
351	1.8615	3.7218	3.1140	0.7627	– 0.1485	– 0.0357
360	– 1.6920	+ 3.7277	– 2.9661	+ 0.7253	0.0000	0.0000

Eleven Contacts.

a	3d Point.		4th Point.		5th Point.	
	x	*y*	*x*	*y*	*x*	*y*
0°	− 2.9661	+ 0.7253	0.0000	0.0000	+ 2.9661	− 0.7253
9	2.8181	0.6880	+ 0.1485	+ 0.0357	3.1140	0.7627
18	2.6701	0.6509	0.2969	0.0715	3.2618	0.8004
27	2.5220	0.6140	0.4454	0.1073	3.4096	0.8382
36	2.3739	0.5772	0.5939	0.1431	3.5573	0.8762
45	2.2257	0.5405	0.7423	0.1789	3.7050	0.9145
54	2.0775	0.5039	0.8907	0.2148	3.8525	0.9530
63	1.9292	0.4675	1.0392	0.2507	4.0000	0.9918
72	1.7810	0.4311	1.1876	0.2866	4.1474	1.0308
81	1.6327	0.3949	1.3360	0.3226	4.2947	1.0702
90	1.4843	0.3587	1.4843	0.3587	4.4419	1.1098
99	1.3360	0.3226	1.6327	0.3949	4.5890	1.1498
108	1.1876	0.2866	1.7810	0.4311	4.7359	1.1902
117	1.0392	0.2507	1.9292	0.4675	4.8827	1.2309
126	0.8907	0.2168	2.0775	0.5039	5.0294	1.2721
135	0.7423	0.1789	2.2257	0.5405	5.1760	1.3137
144	0.5939	0.1431	2.3739	0.5772	5.3223	1.3557
153	0.4454	0.1073	2.5220	0.6140	5.4686	1.3983
162	0.2969	0.0715	2.6701	0.6509	5.6146	1.4414
171	− 0.1485	+ 0.0357	2.8181	0.6880	5.7604	1.4852
180	0.0000	0.0000	2.9661	0.7253	5.9060	1.5295
189	+ 0.1485	− 0.0357	3.1140	0.7627	6.0514	1.5746
198	0.2969	0.0715	3.2618	0.8004	6.1966	1.6204
207	0.4454	0.1073	3.4096	0.8382	6.3414	1.6670
216	0.5939	0.1431	3.5573	0.8762	6.4860	1.7146
225	0.7423	0.1789	3.7050	0.9145	6.6302	1.7631
234	0.8907	0.2148	3.8525	0.9530	6.7740	1.8128
243	1.0392	0.2507	4.0000	0.9918	6.9174	1.8637
252	1.1876	0.2866	4.1474	1.0308	7.0603	1.9160
261	1.3360	0.3226	4.2947	1.0702	7.2027	1.9699
270	1.4843	0.3857	4.4419	1.1098	7.3445	2.0256
279	1.6327	0.3949	4.5890	1.1498	7.4856	2.0834
288	1.7810	0.4311	4.7359	1.1902	7.6258	2.1437
297	1.9292	0.4675	4.8827	1.2309	7.7650	2.2070
306	2.0775	0.5039	5.0294	1.2721	7.9029	2.2739
315	2.2257	0.5405	5.1760	1.3137	8.0393	2.3454
324	2.3739	0.5772	5.3223	1.3557	8.1735	2.4231
333	2.5220	0.6140	5.4686	1.3983	8.3048	2.5094
342	2.6701	0.6509	5.6146	1.4414	8.4313	2.6092
351	2.8181	0.6880	5.7604	1.4852	+ 8.5488	− 2.7353
360	+ 2.9661	− 0.7253	+ 5.9060	+ 1.5295	± 8.6104	∓ 3.0207

TABLE VII.

Eleven Contacts.						
	6th Point.		7th Point.		8th Point.	
a	*x*	*y*	*x*	*y*	*x*	*y*
0°	+ 5.9060	+ 1.5295	± 8.6104	∓ 3.0207	− 5.0232	− 3.9542
9	6.0514	1.5746	− 8.3575	+ 3.2796	4.8542	3.9691
18	6.1966	1.6204	8.1604	3.3789	4.6854	3.9832
27	6.3414	1.6670	7.9724	3.4519	4.5167	3.9966
36	6.4860	1.7146	7.7891	3.5112	4.3483	4.0092
45	6.6302	1.7631	7.6088	3.5618	4.1800	4.0211
54	6.7740	1.8128	7.4306	3.6061	4.0118	4.0324
63	6.9174	1.8637	7.2540	3.6456	3.8438	4.0430
72	7.0603	1.9160	7.0787	3.6814	3.6759	4.0530
81	7.2027	1.9699	6.9043	3.7140	3.5081	4.0625
90	7.3445	2.0256	6.7308	3.7440	3.3404	4.0714
99	7.4856	2.0834	6.5580	3.7718	3.1729	4.0798
108	7.6258	2.1437	6.3859	3.7977	3.0054	4.0877
117	7.7650	2.2070	6.2142	3.8218	2.8380	4.0951
126	7.9029	2.2739	6.0430	3.8443	2.6707	4.1019
135	8.0393	2.3454	5.8722	3.8655	2.5034	4.1083
144	8.1735	2.4231	5.7018	3.8854	2.3363	4.1143
153	8.3048	2.5094	5.5318	3.9041	2.1691	4.1198
162	8.4313	2.6092	5.3620	3.9218	2.0021	4.1248
171	+ 8.5488	+ 2.7353	5.1925	3.9385	1.8351	4.1294
180	± 8.6104	± 3.0207	5.0232	3.9542	1.6681	4.1336
189	− 8.5488	− 2.7353	4.8542	3.9691	1.5012	4.1374
198	8.4313	2.6092	4.6854	3.9832	1.3343	4.1408
207	8.3048	2.5094	4.5167	3.9966	1.1674	4.1437
216	8.1735	2.4231	4.3483	4.0092	1.0006	4.1463
225	8.0393	2.3454	4.1800	4.0211	0.8338	4.1484
234	7.9029	2.2739	4.0118	4.0324	0.6670	4.1502
243	7.7650	2.2070	3.8438	4.0430	0.5003	4.1515
252	7.6258	2.1437	3.6759	4.0530	0.3335	4.1525
261	7.4858	2.0834	3.5081	4.0625	− 0.1667	4.1531
270	7.3445	2.0256	3.3404	4.0714	0.0000	4.1533
279	7.2027	1.9699	3.1729	4.0798	+ 0.1667	4.1531
288	7.0603	1.9160	3.0054	4.0877	0.3335	4.1525
297	6.9174	1.8637	2.8380	4.0951	0.5003	4.1515
306	6.7740	1.8128	2.6707	4.1019	0.6670	4.1502
315	6.6302	1.7631	2.5034	4.1083	0.8338	4.1484
324	6.4860	1.7146	2.3363	4.1143	1.0006	4.1463
333	6.3414	1.6670	2.1691	4.1198	1.1674	4.1437
342	6.1966	1.6204	2.0021	4.1248	1.3343	4.1408
351	6.0514	1.5746	1.8351	4.1294	1.5012	4.1374
360	− 5.9060	− 1.5295	− 1.6681	+ 4.1336	+ 1.6681	− 4.1336

TABLE VII.

Eleven Contacts.

α	9th Point.		10th Point.		11th Point.	
	x	y	x	y	x	y
0°	− 1.6681	+ 4.1334	+ 1.6681	− 4.1336	+ 5.0232	+ 3.9542
9	1.5012	4.1374	1.8351	4.1294	5.1925	3.9385
18	1.3343	4.1408	2.0021	4.1248	5.3620	3.9218
27	1.1674	4.1437	2.1691	4.1198	5.5318	3.9041
36	1.0006	4.1463	2.3363	4.1143	5.7018	3.8854
45	0.8338	4.1484	2.5034	4.1083	5.8722	3.8655
54	0.6670	4.1502	2.6707	4.1019	6.0430	3.8443
63	0.5003	4.1515	2.8380	4.0951	6.2142	3.8218
72	0.3335	4.1525	3.0054	4.0877	6.3859	3.7977
81	− 0.1667	4.1531	3.1729	4.0798	6.5580	3.7718
90	0.0000	4.1533	3.3404	4.0714	6.7308	3.7440
99	+ 0.1667	4.1531	3.5081	4.0625	6.9043	3.7140
108	0.3335	4.1525	3.6759	4.0530	7.0787	3.6814
117	0.5003	4.1515	3.8438	4.0430	7.2540	3.6456
126	0.6670	4.1502	4.0118	4.0324	7.4306	3.6061
135	0.8338	4.1484	4.1800	4.0211	7.6088	3.5618
144	1.0006	4.1463	4.3483	4.0092	7.7891	3.5112
153	1.1674	4.1437	4.5167	3.9966	7.9724	3.4519
162	1.3343	4.1408	4.6854	3.9832	8.1604	3.3789
171	1.5012	4.1374	4.8542	3.9691	+ 8.3575	+ 3.2796
180	1.6681	4.1336	5.0232	3.9542	± 8.6104	± 3.0207
189	1.8351	4.1294	5.1925	3.9385	− 8.3575	− 3.2796
198	2.0021	4.1248	5.3620	3.9218	8.1604	3.3789
207	2.1691	4.1198	5.5318	3.9041	7.9724	3.4519
216	2.3363	4.1143	5.7018	3.8854	7.7891	3.5112
225	2.5034	4.1083	5.8722	3.8655	7.6088	3.5618
234	2.6707	4.1019	6.0430	3.8443	7.4306	3.6061
243	2.8380	4.0951	6.2142	3.8218	7.2540	3.6456
252	3.0054	4.0877	6.3859	3.7977	7.0787	3.6814
261	3.1729	4.0798	6.5580	3.7718	6.9043	3.7140
270	3.3404	4.0714	6.7308	3.7440	6.7308	3.7440
279	3.5081	4.0625	6.9043	3.7140	6.5580	3.7718
288	3.6759	4.0530	7.0787	3.6814	6.3859	3.7977
297	3.8438	4.0430	7.2540	3.6456	6.2142	3.8218
306	4.0118	4.0324	7.4306	3.6061	6.0430	3.8443
315	4.1800	4.0211	7.6088	3.5618	5.8722	3.8655
324	4.3483	4.0092	7.7891	3.5112	5.7018	3.8854
333	4.5167	3.9966	7.9724	3.4519	5.5318	3.9041
342	4.6854	3.9832	8.1604	3.3789	5.3620	3.9218
351	4.8542	3.9691	+ 8.3575	− 3.2796	5.1925	3.9385
360	+ 5.0232	+ 3.9542	± 8.6104	± 3.0207	− 5.0232	− 3.9542

TABLE VIII.

TRUNCATED INVOLUTES.

n	X	Y	i	QM	MB	QB
			° ′			
1						
3	2.150	1.460	34··11	4.626	6.454	1.829
5	3.847	1.824	25··22	9.938	12.375	2.438
7	5.473	2.106	21··03	16.329	19.230	2.901
9	7.074	2.349	18··22	23.656	26.949	3.294
11	8.664	2.567	16··30	31.814	35.456	3.642
13	10.249	2.767	15··06	40.729	44.687	3.958
15	11.829	2.953	14··01	50.341	54.591	4.250
17	13.408	3.128	13··08	60.604	65.127	4.523
19	14.984	3.293	12··24	71.479	76.434	4.955
21	16.560	3.450	11··46	82.934	87.957	5.023
23	18.135	3.601	11··14	94.940	100.195	5.255
NUMBER OF POINTS IN CONTACT.	FOR THE CORNER OF THE ORBIT.	FOR THE CORNER OF THE ORBIT.	OBLIQUITY OF PRAGMATIC ACTION.	LIMIT OF COMPLETE CONTACTS.	RADIUS OF SUPPLEMENTARY ARC.	HALF DEPTH OF TOOTH.

TABLE IX.

Truncated Epicycloids.

n	X	Y	i	QM	MB	QB
			° ′			
1						
3	2.598	1.500	30··00	3.000	5.196	2.196
5	4.300	1.654	21··02	6.418	9.146	2.728
7	5.919	1.814	17··02	10.565	13.721	3.156
9	7.515	1.968	14··40	15.333	18.863	3.530
11	9.101	2.114	13··04	20.650	24.516	3.866
13	10.682	2.251	11··54	26.466	30.640	4.174
15	12.260	2.382	11··00	32.740	37.200	4.461
17	13.836	2.507	10··16	39.441	44.171	4.730
19	15.404	2.626	9··40	46.545	51.506	4.961
21	16.986	2.739	9··10	54.029	59.255	5.226
23	18.559	2.849	8··44	61.874	67.332	5.457
NUMBER OF POINTS IN CONTACT.	FOR THE CORNER OF THE ORBIT.	FOR THE CORNER OF THE ORBIT.	GREATEST OBLIQUITY OF PRAGMATIC ACTION.	LIMIT OF COMPLETE CONTACTS. RADIUS OF PRAGMATIC ARC.	RADIUS OF PLEROTIC ARC.	HALF DEPTH OF TOOTH.

TABLE X.

TABLE XI.

Kemend.			Seven Contacts. QN = QM.			
	General.		1st Point.		2d Point.	
a	x	y	x	y	x	y
0°	0.0000	0.0000	± 5·8940	± 1.8999	− 3.0730	+ 0.4760
9	+ 0.0952	+ 0.0767	− 5.8414	− 1.5417	2.9204	0.4472
18	0.1905	0.1534	5.7208	1.3982	2.7675	0.4192
27	0.2861	0.2299	5.5893	1.2907	2.6145	0.3920
36	0.3819	0.3062	5.4522	1.2020	2.4614	0.3656
45	0.4782	0.3821	5.3117	1.1254	2.3081	0.3398
54	0.5751	0.4576	5.1687	1.0575	2.1548	0.3146
63	0.6727	0.5325	5.0240	0.9961	2.0013	0.2900
72	0.7712	0.6068	4.8779	0.9400	1.8477	0.2659
81	0.8707	0.6802	4.7307	0.8881	1.6940	0.2423
90	0.9715	0.7525	4.5826	0.8399	1.5402	0.2190
99	1.0739	0.8235	4.4337	0.7946	1.3864	0.1961
108	1.1780	0.8929	4.2841	0.7521	1.2325	0.1735
117	1.2844	0.9603	4.1341	0.7118	1.0785	0.1513
126	1.3935	1.0249	3.9835	0.6735	0.9245	0.1292
135	1.5061	1.0860	3.8325	0.6371	0.7705	0.1074
144	1.6231	1.1422	3.6812	0.6022	0.6164	0.0857
153	1.7464	1.1909	3.5296	0.5688	0.4624	0.0641
162	1.8790	1.2275	3.3776	0.5368	0.3082	0.0427
171	+ 2.0285	+ 1.2392	3.2254	0.5059	− 0.1541	+ 0.0213
180	± 2.2667	± 1.0861	3.0730	0.4760	0.0000	0.0000
189	− 2.2772	− 0.6407	2.9204	0.4472	+ 0.1541	− 0.0213
198	2.1834	0.3566	2.7675	0.4192	0.3082	0.0427
207	2.0540	− 0.0942	2.6145	0.3920	0.4624	0.0641
216	1.8926	+ 0.1582	2.4614	0.3656	0.6164	0.0857
225	1.6969	0.4034	2.3081	0.3398	0.7705	0.1074
234	1.4614	0.6396	2.1548	0.3146	0.9245	0.1292
243	1.1783	0.8605	2.0013	0.2900	1.0785	0.1513
252	0.8390	1.0520	1.8477	0.2659	1.2325	0.1735
261	− 0.4406	1.1891	1.6940	0.2423	1.3864	0.1961
270	0.0000	1.2402	1.5402	0.2190	1.5402	0.2190
279	+ 0.4406	1.1891	1.3864	0.1961	1.6940	0.2423
288	0.8390	1.0520	1.2325	0.1735	1.8477	0.2659
297	1.1783	0.8605	1.0785	0.1503	2.0013	0.2900
306	1.4614	0.6396	0.9245	0.1292	2.1548	0.3146
315	1.6969	0.4034	0.7705	0.1074	2.3081	0.3398
324	1.8926	+ 0.1582	0.6164	0.0857	2.4614	0.3656
333	2.0540	− 0.0942	0.4624	0.0641	2.6145	0.3920
342	2.1834	0.3566	0.3082	0.0427	2.7675	0.4192
351	+ 2.2772	− 0.6407	− 0.1541	− 0.0213	2.9204	0.4472
360	± 2.2667	∓ 1.0861	0.0000	0.0000	+ 3.0730	− 0.4760

TABLE XI.

SEVEN CONTACTS. $QN = QM$.

	3d Point.		4th Point.		5th Point.	
a	x	y	x	y	x	y
0°	0.0000	0.0000	+ 3.0730	− 0.4760	± 5.8940	± 1.8999
9	+ 0.1541	+ 0.0213	3·2254	0.5059	− 5.6053	− 2.2647
18	0.3082	0.0427	3.3776	0.5368	5.3847	2.4139
27	0.4624	0.0641	3.5296	0.5688	5.1750	2.5265
36	0.6164	0.0857	3.6812	0.6022	4.9708	2.6194
45	0.7705	0.1074	3.8325	0.6371	4.7701	2.6994
54	0.9245	0.1292	3.9835	0.6735	4.5717	2.7700
63	1.0785	0.1513	4.1341	0.7118	4.3751	2.8331
72	1.2325	0.1735	4.2841	0.7521	4.1798	2.8902
81	1.3864	0.1961	4.4337	0.7946	3.9857	2.9421
90	1.5402	0.2190	4.5826	0.8399	3.7924	2.9895
99	1.6940	0.2423	4.7307	0.8881	3.5998	3.0331
108	1.8477	0.2659	4.8779	0.9400	3.4079	3.0731
117	2.0013	0.2900	5.0240	0.9961	3.2165	3.1098
126	2.1548	0.3146	5.1687	1.0575	3.0256	3.1436
135	2.3081	0.3398	5.3117	1.1254	2.8350	3.1747
144	2.4614	0.3656	5.4522	1.2020	2.6448	3.2032
153	2.6145	0.3920	5.5893	1.2907	2.4549	3.2292
162	2.7675	0.4192	5.7208	1.3982	2.2652	3.2529
171	2.9204	0.4472	+ 5.8414	− 1.5417	2.0758	3.2745
180	3.0730	0.4760	± 5.8940	∓ 1.8999	1.8865	3.2939
189	3.2254	0.5059	− 5.8414	+ 1.5417	1.6974	3.3113
198	3.3776	0.5368	5.7208	1.3982	1.5085	3.3266
207	3.5296	0.5688	5.5893	1.2907	1.3197	3.3401
216	3.6812	0.6022	5.4522	1.2020	1.1309	3.3517
225	3.8325	0.6371	5.3117	1.1254	0.9423	3.3614
234	3.9835	0.6735	5.1687	1.0575	0.7538	3.3694
243	4.1341	0.7118	5.0240	0.9961	0.5653	3.3755
252	4.2841	0.7521	4.8779	0.9400	0.3768	3.3799
261	4.4337	0.7946	4.7307	0.8881	− 0.1884	3.3825
270	4.5826	0.8399	4.5826	0.8399	0.0000	3.3834
279	4.7307	0.8881	4.4337	0.7946	+ 0.1884	3.3825
288	4.8779	0.9400	4.2841	0.7521	0.3768	3.3799
297	5.0240	0.9961	4.1341	0.7118	0.5653	3.3755
306	5.1687	1.0575	3.9835	0.6735	0.7538	3.3694
315	5.3117	1.1254	3.8325	0.6371	0.9423	3.3614
324	5.4522	1.2020	3.6812	0.6022	1.1309	3.3517
333	5.5893	1.2907	3.5296	0.5688	1.3197	3.3401
342	5.7208	1.3982	3.3776	0.5368	1.5085	3.3266
351	+ 5.8414	+ 1.5417	3.2254	0.5059	1.6974	3.3113
360	± 5.8940	± 1.8999	− 3.0730	+ 0.4760	+ 1.8865	− 3.2939

	Seven Contacts. QN = QM.				QN = QO.	
	6th Point.		7th Point.		1st Point.	
α	x	y	x	y	x	y
0°	– 1.8865	+ 3.2939	+ 1.8865	– 3.2939	± 5.7079	± 2.1737
9	1.6974	3.3113	2.0758	3.2745	– 5.6655	– 1.8479
18	1.5085	3.3266	2.2652	3.2529	5.5519	1.7108
27	1.3197	3.3401	2.4549	3.2292	5.4267	1.6052
36	1.1309	3.3517	2.6448	3.2032	5.2956	1.5158
45	0.9423	3.3614	2.8350	3.1747	5.1608	1.4369
54	0.7538	3.3694	3.0256	3.1436	5.0234	1.3655
63	0.5653	3.3755	3.2165	3.1098	4.8839	1.2998
72	0.3768	3.3799	3.4079	3.0731	4.7430	1.2385
81	– 0.1884	3.3825	3.5998	3.0331	4.6009	1.1808
90	0.0000	3.3834	3.7924	2.9895	4.4577	1.1262
99	+ 0.1884	3.3825	3.9857	2.9421	4.3137	1.0742
108	0.3768	3.3799	4.1798	2.8902	4.1689	1.0245
117	0.5653	3.3755	4.3751	2.8331	4.0235	0.9766
126	0.7538	3.3694	4.5717	2.7700	3.8775	0.9279
135	0.9423	3.3614	4.7701	2.6994	3.7311	0.8859
144	1.1309	3.3517	4.9708	2.6194	3.5842	0.8427
153	1.3197	3.3401	5.1750	2.5265	3.4370	0.8007
162	1.5085	3.3266	5.3847	2.4139	3.2894	0.7597
171	1.6974	3.3113	+ 5.6053	– 2.2647	3.1415	0.7198
180	1.8865	3.2939	± 5.8940	∓ 1.8999	2.9934	0.6807
189	2.0758	3.2745	– 5.6053	+ 2.2647	2·8449	0.6425
198	2.2652	3.2529	5.3847	2.4139	2.6963	0.6050
207	2.4549	3.2292	5.1750	2.5265	2.5475	0.5682
216	2.6448	3.2032	4.9708	2.6194	2.3984	0.5319
225	2.8350	3.1747	4.7701	2.6994	2.2493	0.4963
234	3.0256	3.1436	4.5717	2.7700	2.0999	0.4611
243	3.2165	3.1098	4.3751	2.8331	1.9504	0.4299
252	3.4079	3.0731	4.1798	2.8902	1.8008	0.3921
261	3.5998	3.0331	3.9857	2.9421	1.6511	0.3582
270	3.7924	2.9895	3.7924	2.9895	1.5013	0.3256
279	3.9857	2.9421	3.5998	3.0331	1.3514	0.2913
288	4.1798	2.8902	3.4079	3.0731	1.2015	0.2583
297	4.3751	2.8331	3.2165	3.1098	1.0514	0.2255
306	4.5714	2.7700	3.0256	3.1436	0.9013	0.1930
315	4.7701	2.6994	2.8350	3.1747	0.7512	0.1605
324	4.9708	2.6194	2.6448	3.2032	0.6010	0.1283
333	5.1750	2.5265	2.4549	3.2292	0.4508	0.0961
342	5.3847	2.4139	2.2652	3.2529	0.3005	0.0640
351	+ 5.6053	+ 2.2647	2.0758	3.2745	– 0.1513	– 0.0320
360	± 5.8940	± 1.8999	– 1.8865	+ 3.2939	0.0000	0.0000

TABLE XII.

Seven Contacts,					QN = QO.	
	2d Point.		3d Point.		4th Point.	
a	x	y	x	y	x	y
0°	− 2.9934	+ 0.6807	0.0000	0.0000	+ 2.9934	− 0.6807
9	2.8449	0.6425	+ 0.1503	+ 0.0320	3.1415	0.7198
18	2.6963	0.6050	0.3005	0.0640	3.2894	0.7597
27	2.5475	0.5682	0.4508	0.0961	3.4370	0.8007
36	2.3984	0.5319	0.6010	0.1283	3.5842	0.8427
45	2.2493	0.4963	0.7512	0.1605	3.7311	0.8859
54	2.0999	0.4611	0.9013	0.1930	3.8775	0.9279
63	1.9504	0.4299	1.0514	0.2255	4.0235	0.9766
72	1.8008	0.3921	1.2015	0.2583	4.1689	1.0245
81	1.6511	0.3582	1.3514	0.2913	4.3137	1.0742
90	1.5013	0.3256	1.5013	0.3256	4.4577	1.1262
99	1.3514	0.2913	1.6511	0.3582	4.6009	1.1808
108	1.2015	0.2583	1.8008	0.3921	4.7430	1.2385
117	1.0514	0.2255	1.9504	0.4299	4.8839	1.2998
126	0.9013	0.1930	2.0999	0.4611	5.0234	1.3655
135	0.7512	0.1605	2.2493	0.4963	5.1608	1.4369
144	0.6010	0.1283	2.3984	0.5319	5.2956	1.5158
153	0.4508	0.0961	2.5475	0.5682	5.4267	1.6052
162	0.3005	0.0640	2.6963	0.6050	5.5519	1.7108
171	− 0.1503	+ 0.0320	2.8449	0.6425	+ 5.6655	− 1.8479
180	0.0000	0.0000	2.9934	0.6807	± 5.7079	∓ 2.1737
189	+ 0.1503	− 0.0320	3.1415	0·7198	− 5.6655	+ 1.8479
198	0.3005	0.0640	3.2894	0.7597	5.5519	1.7108
207	0.4508	0.0961	3.4370	0.8007	5.4267	1.6052
216	0.6010	0.1283	3.5842	0.8427	5.2956	1.5158
225	0.7512	0.1605	3.7311	0.8859	5.1608	1.4369
234	0.9013	0.1930	3.8775	0.9279	5.0234	1.3655
243	1.0514	0.2255	4.0235	0.9766	4.8839	1.2998
252	1.2015	0.2583	4.1689	1.0245	4.7430	1.2385
261	1.3514	0.2913	4.3137	1.0742	4.6009	1.1808
270	1.5013	0.3256	4.4577	1.1262	4.4577	1.1262
279	1.6511	0.3582	4.6009	1.1808	4.3137	1.0742
288	1.8008	0.3921	4.7430	1.2385	4.1689	1.0245
297	1.9504	0.4299	4.8839	1.2998	4.0235	0.9766
306	2.0999	0.4611	5.0234	1.3655	3.8775	0.9279
315	2.2493	0.4963	5.1608	1.4369	3.7311	0.8859
324	2.3984	0.5319	5.2956	1.5158	3.5842	0.8427
333	2.5475	0.5682	5.4267	1.6052	3.4370	0.8007
342	2.6963	0.6050	5.5519	1.7108	3.2894	0.7597
351	2.8449	0.6425	+ 5.6655	+ 1.8479	3.1415	0.7198
360	+ 2.9934	− 0.6807	± 5.7079	± 2.1737	− 2.9934	+ 0.6807

TABLE XII.

Seven Contacts. QN = QO.

	5th Point.		6th Point.		7th Point.	
a	x	y	x	y	x	y
0°	± 5.7079	± 2.1737	− 1.8170	+ 3.3020	+ 1.8170	− 3.3020
9	− 5.4204	− 2.4867	1.6348	3.3152	1.9994	3.2874
18	5.2040	2.6104	1.4527	3.3268	2.1821	3.2710
27	4.9991	2.7023	1.2708	3.3369	2.3650	3.2530
36	4.8000	2.7773	1·0891	3.3456	2.5483	3.2332
45	4.6047	2.8413	0.9074	3.3530	2.7318	3.2115
54	4.4120	2.8973	0.7258	3.3589	2.9158	3.1878
63	4.2213	2.9471	0.5443	3.3635	3.1002	3.1619
72	4.0320	2.9918	0.3628	3.3668	3.2851	3.1336
81	3.8439	3.0323	− 0.1814	3.3688	3.4707	3.1028
90	3.6569	3.0692	0.0000	3.3694	3.6569	3.0692
99	3.4707	3.1028	+ 0.1814	3.3688	3.8439	3.0323
108	3.2851	3.1336	0.3628	3.3668	4.0320	2.9918
117	3.1002	3.1619	0.5443	3.3635	4.2213	2.9471
126	2.9158	3.1878	0.7258	3.3589	4.4120	2.8973
135	2.7318	3.2115	0.9074	3.3530	4.6047	2.8413
144	2.5483	3.2332	1.0891	3.3456	4.8000	2.7773
153	2.3650	3.2530	1.2708	3.3369	4.9991	2.7023
162	2.1821	3.2710	1.4527	3.3268	5.2040	2.6104
171	1.9994	3.2874	1.6348	3.3152	+ 5.4204	− 2.4867
180	1.8170	3.3020	1·8170	3.3020	± 5.7079	∓ 2.1737
189	1.6348	3.3152	1.9994	3.2874	− 5.4204	+ 2.4867
198	1.4527	3.3268	2.1821	3.2710	5.2040	2.6104
207	1.2708	3.3369	2.3650	3.2530	4.9991	2.7023
216	1.0891	3.3456	2.5483	3.2332	4.8000	2.7773
225	0.9074	3.3530	2.7318	3.2115	4.6047	2.8413
234	0.7258	3.3589	2.9158	3.1878	4.4120	2.8973
243	0.5443	3.3635	3.1002	3.1619	4.2213	2.9471
252	0.3628	3.3668	3.2851	3.1336	4.0320	2.9918
261	− 0.1814	3.3688	3.4707	3.1028	3.8439	3.0323
270	0.0000	3.3694	3.6569	3.0692	3.6569	3.0692
279	+ 0.1814	3.3688	3·8439	3.0323	3.4707	3.1028
288	0.3628	3.3668	4.0320	2.9918	3.2851	3.1336
297	0.5443	3.3635	4.2213	2.9471	3.1002	3.1619
306	0.7258	3.3589	4.4120	2.8973	2.9158	3.1878
315	0.9074	3.3530	4.6047	2.8413	2.7318	3.2115
324	1.0891	3.3456	4 8000	2.7773	2.5483	3.2332
333	1.2708	3.3369	4.9991	2.7023	2.3650	3.2530
342	1.4527	3.3268	5.2040	2.6104	2.1821	3.2710
351	1.6348	3.3152	+ 5.4204	+ 2.4867	1.9994	3.2874
360	+ 1.8170	− 3.3020	± 5.7079	± 2.1737	− 1.8170	+ 3.3020

Teeth of Wheels. I.

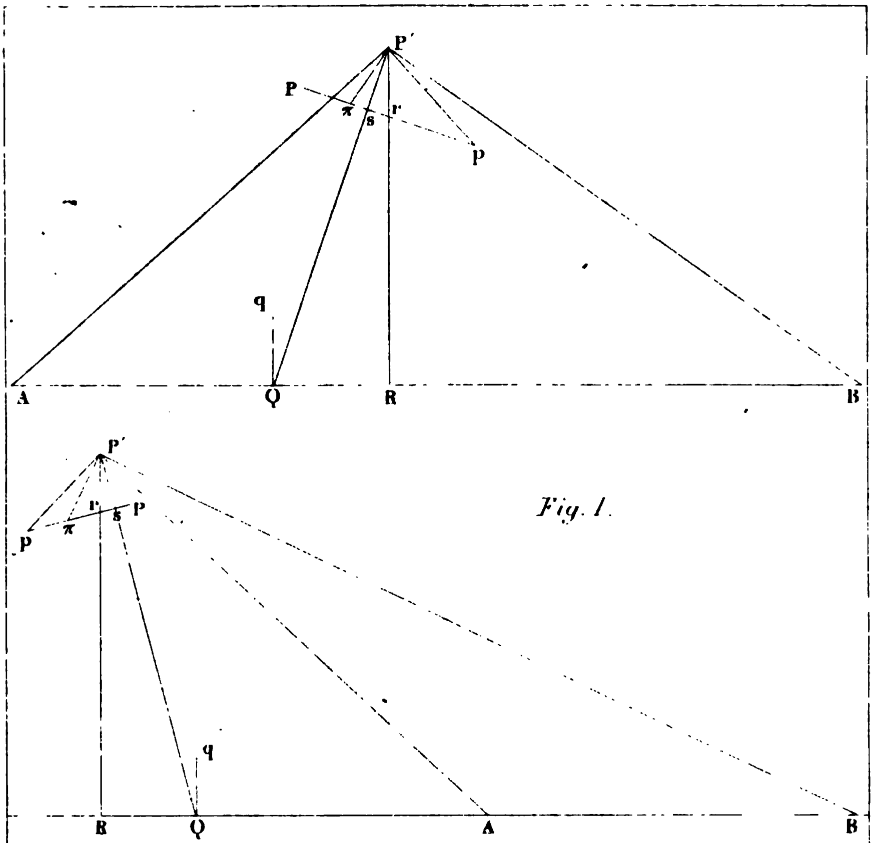


Fig. 1.

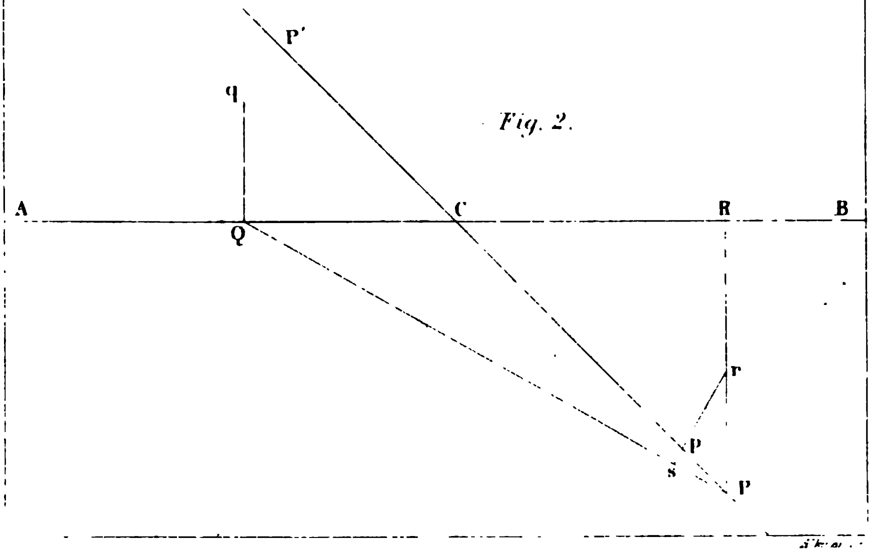


Fig. 2.

TABLE XII.

Seven Contacts. QN = QO.

	5th Point.		6th Point.		7th Point.	
a	*x*	*y*	*x*	*y*	*x*	*y*
° 0	± 5.7079	± 2.1737	− 1.8170	+ 3.3020	+ 1.8170	− 3.3020
9	− 5.4204	− 2.4867	1.6348	3.3152	1.9994	3.2874
18	5.2040	2.6104	1.4527	3.3268	2.1821	3.2710
27	4.9991	2.7023	1.2708	3.3369	2.3650	3.2530
36	4.8000	2.7773	1·0891	3.3456	2.5483	3.2332
45	4.6047	2.8413	0.9074	3.3530	2.7318	3.2115
54	4.4120	2.8973	0.7258	3.3589	2.9158	3.1878
63	4.2213	2.9471	0.5443	3.3635	3.1002	3.1619
72	4.0320	2.9918	0.3628	3.3668	3.2851	3.1336
81	3.8439	3.0323	− 0.1814	3.3688	3.4707	3.1028
90	3.6569	3.0692	0.0000	3.3694	3.6569	3.0692
99	3.4707	3.1028	+ 0.1814	3.3688	3.8439	3.0323
108	3.2851	3.1336	0.3628	3.3668	4.0320	2.9918
117	3.1002	3.1619	0.5443	3.3635	4.2213	2.9471
126	2.9158	3.1878	0.7258	3.3589	4.4120	2.8973
135	2.7318	3.2115	0.9074	3.3530	4.6047	2.8413
144	2.5483	3.2332	1.0891	3.3456	4.8000	2.7773
153	2.3650	3.2530	1.2708	3.3369	4.9991	2.7023
162	2.1821	3.2710	1.4527	3.3268	5.2040	2.6104
171	1.9994	3.2874	1.6348	3.3152	+ 5.4204	− 2.4867
180	1.8170	3.3020	1·8170	3.3020	± 5.7079	∓ 2.1737
189	1.6348	3.3152	1.9994	3.2874	− 5.4204	+ 2.4867
198	1.4527	3.3268	2.1821	3.2710	5.2040	2.6104
207	1.2708	3.3369	2.3650	3.2530	4.9991	2.7023
216	1.0891	3.3456	2.5483	3.2332	4.8000	2.7773
225	0.9074	3.3530	2.7318	3.2115	4.6047	2.8413
234	0.7258	3.3589	2.9158	3.1878	4.4120	2.8973
243	0.5443	3.3635	3.1002	3.1619	4.2213	2.9471
252	0.3628	3.3668	3.2851	3.1336	4.0320	2.9918
261	− 0.1814	3.3688	3.4707	3.1028	3.8439	3.0323
270	0.0000	3.3694	3.6569	3.0692	3.6569	3.0692
279	+ 0.1814	3.3688	3·8439	3.0323	3.4707	3.1028
288	0.3628	3.3668	4.0320	2.9918	3.2851	3.1336
297	0.5443	3.3635	4.2213	2.9471	3.1002	3.1619
306	0.7258	3.3589	4.4120	2.8973	2.9158	3.1878
315	0.9074	3.3530	4.6047	2.8413	2.7318	3.2115
324	1.0891	3.3456	4 8000	2.7773	2.5483	3.2332
333	1.2708	3.3369	4.9991	2.7023	2.3650	3.2530
342	1.4527	3.3268	5.2040	2.6104	2.1821	3.2710
351	1.6348	3.3152	+ 5.4204	+ 2.4867	1.9994	3.2874
360	+ 1.8170	− 3.3020	± 5.7079	± 2.1737	− 1.8170	+ 3.3020

Teeth of Wheels. 1.

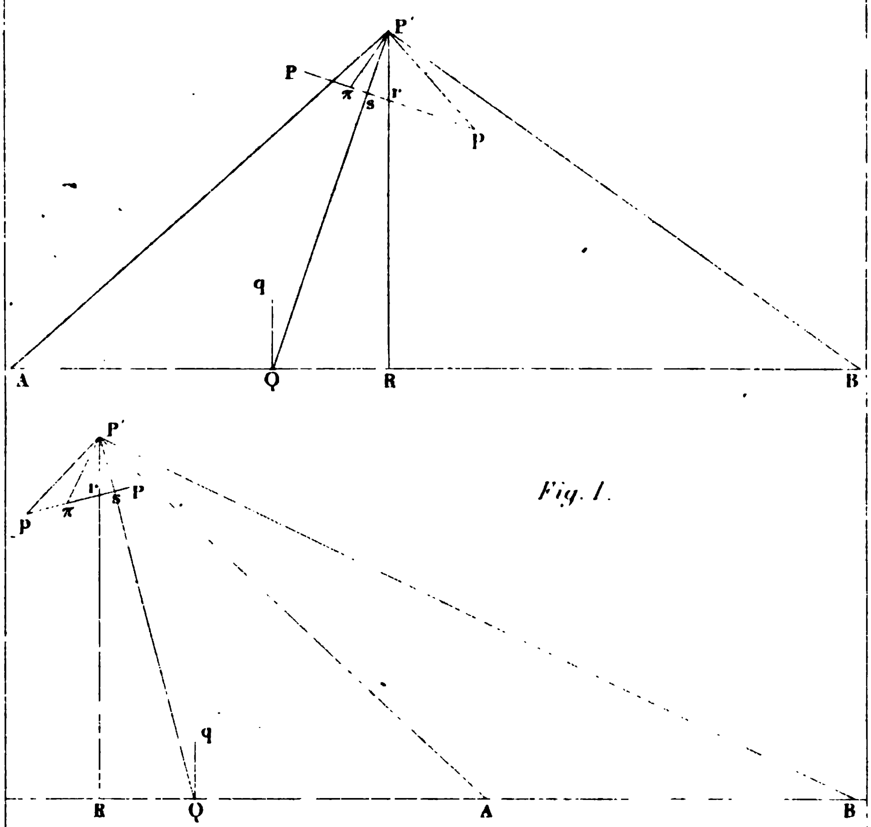


Fig. 1.

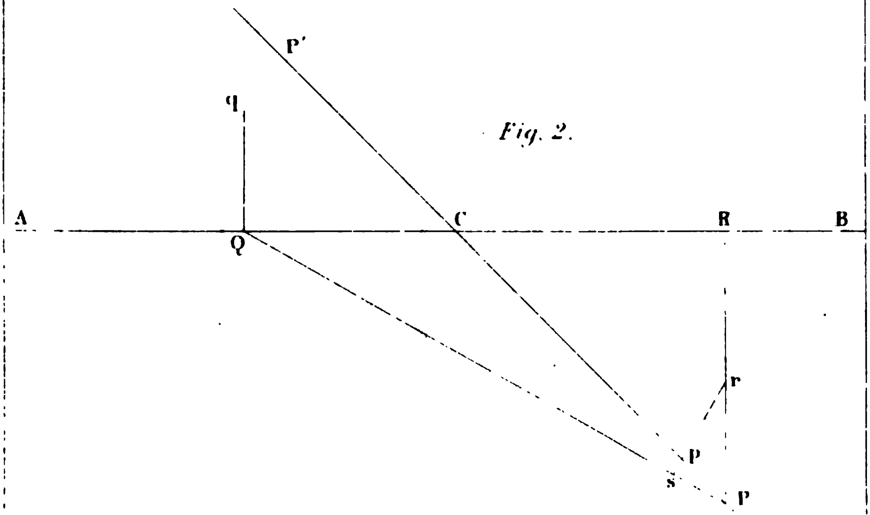


Fig. 2.

Fig. 3.

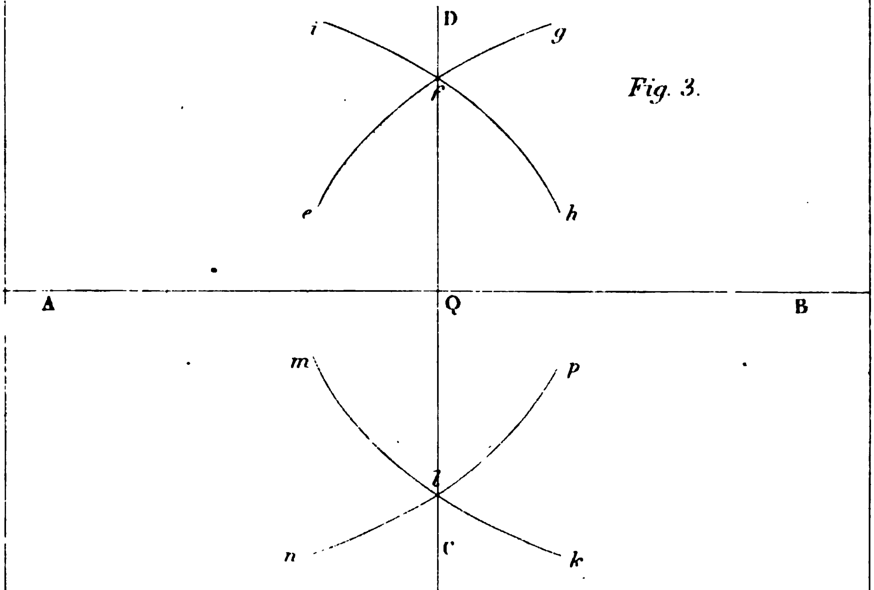


Fig. 4.

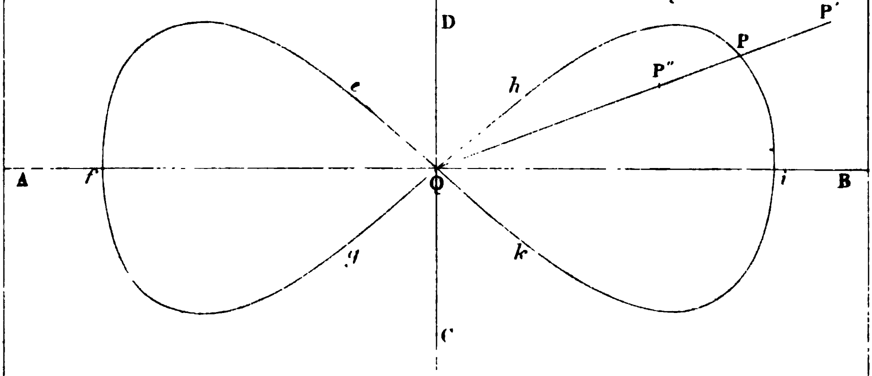


Fig. 26.

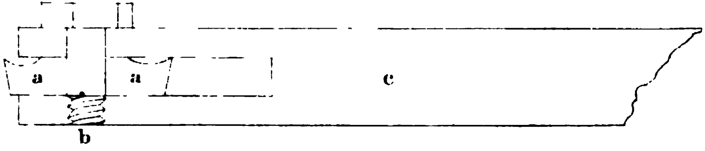


Fig. 5.

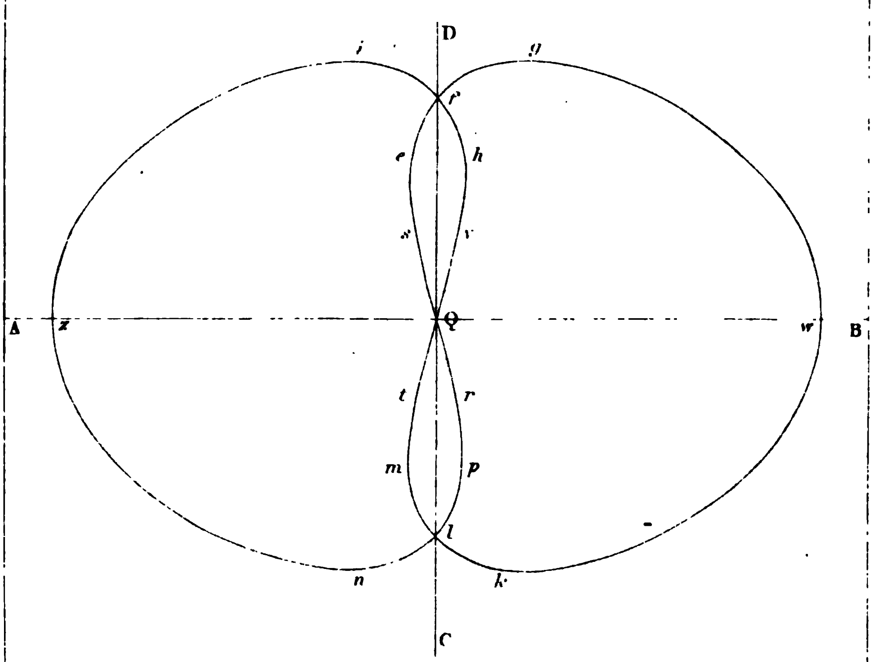
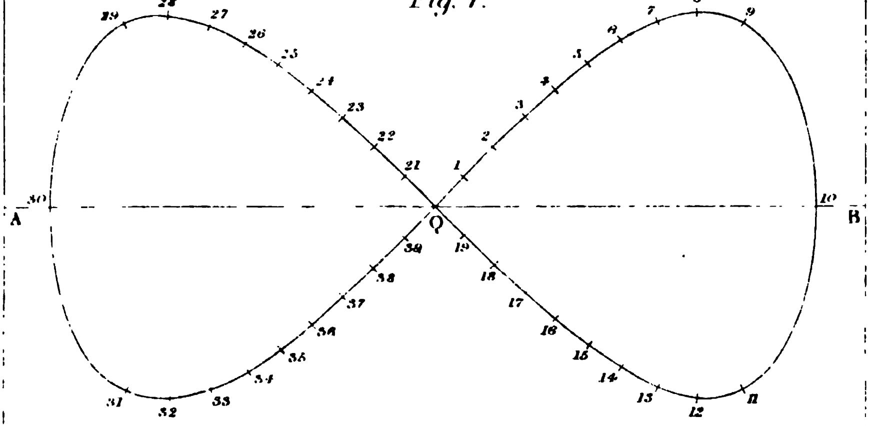

D
i
g
f
e
h
s
v
A
z
Q
w
B
t
r
m
p
l
n
k
C

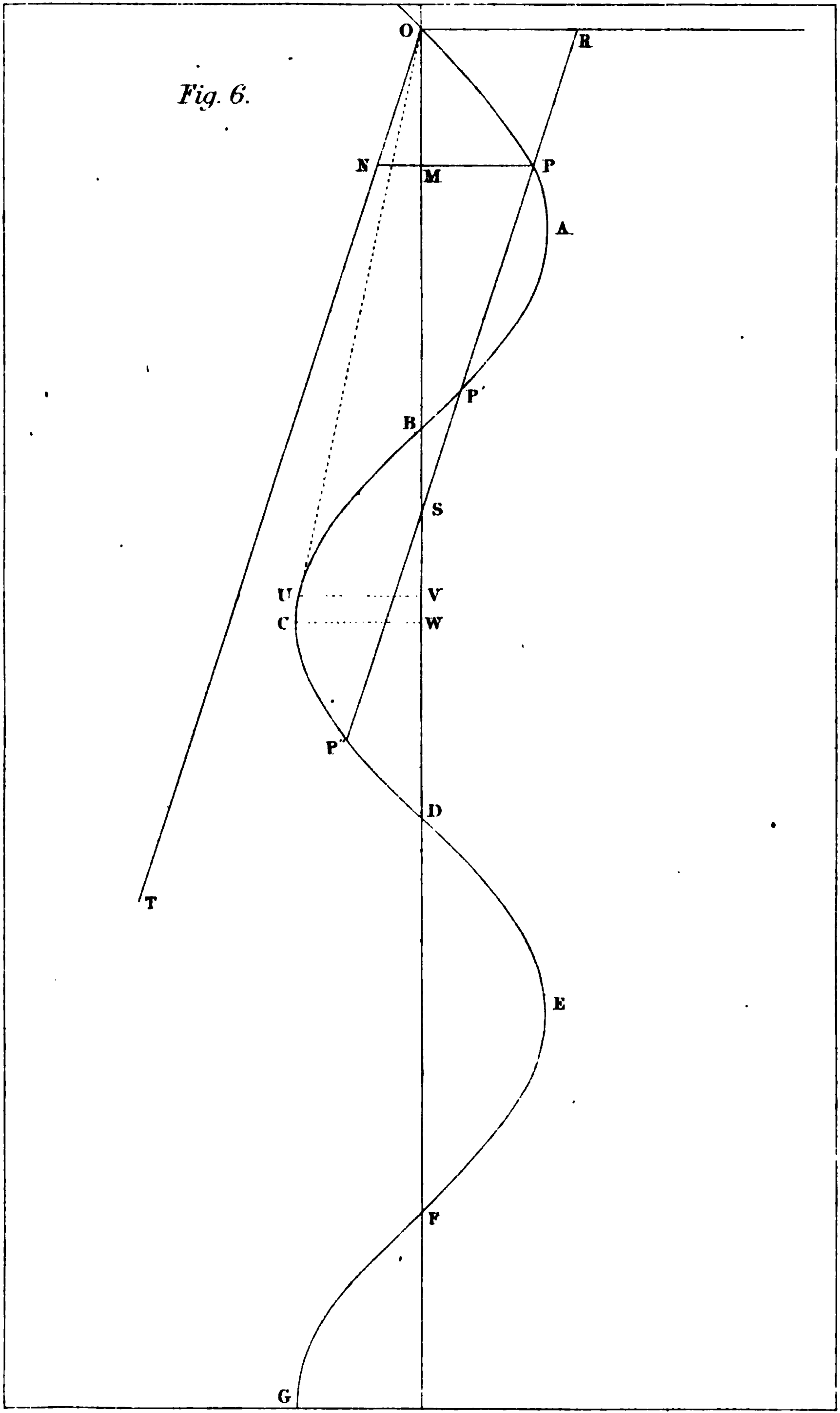
Fig. 7.


A
Q
B

[illegible]

Teeth of Wheels. IV.

Fig. 6.



Aikman sc.

Fig. 8.

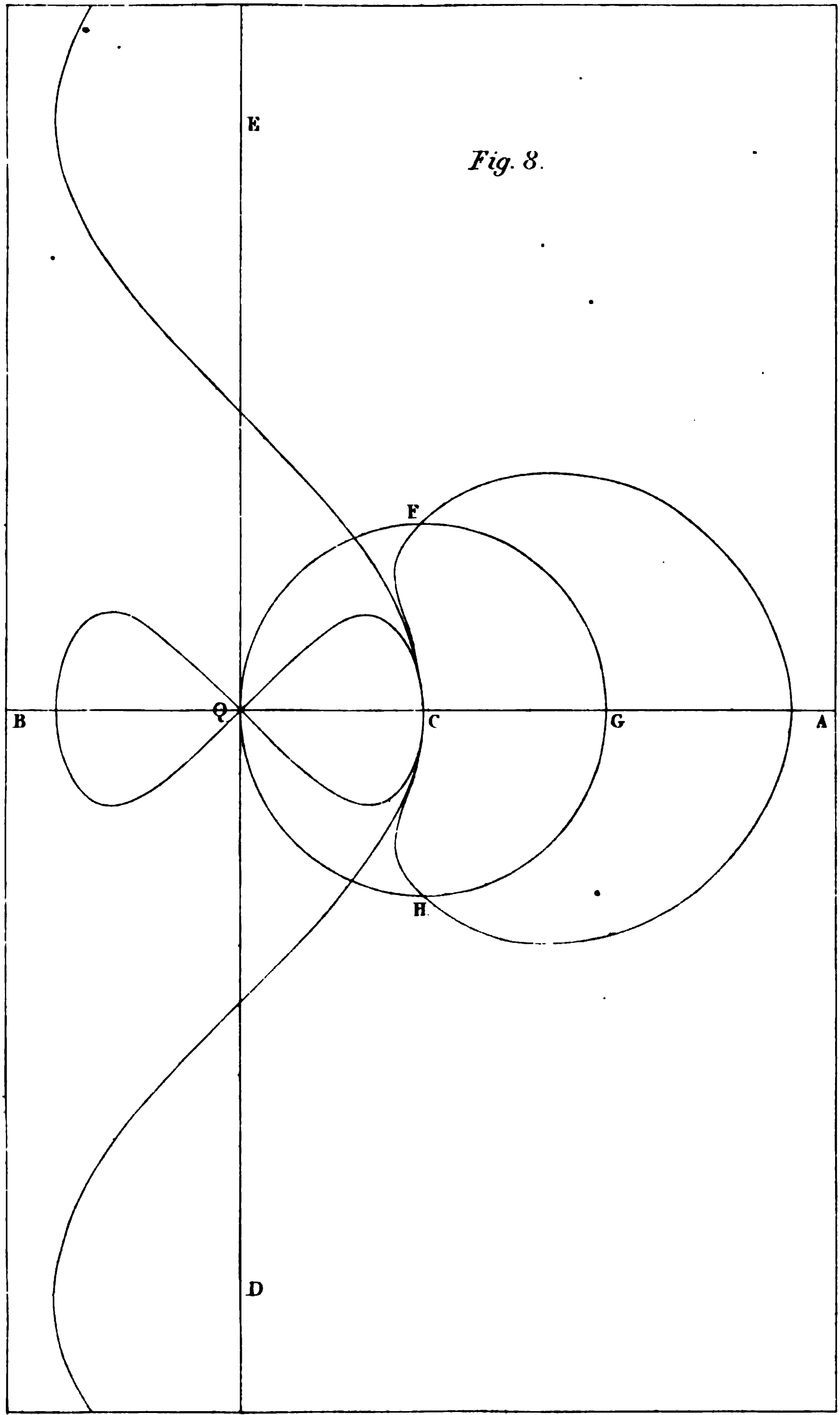


Fig. 9.

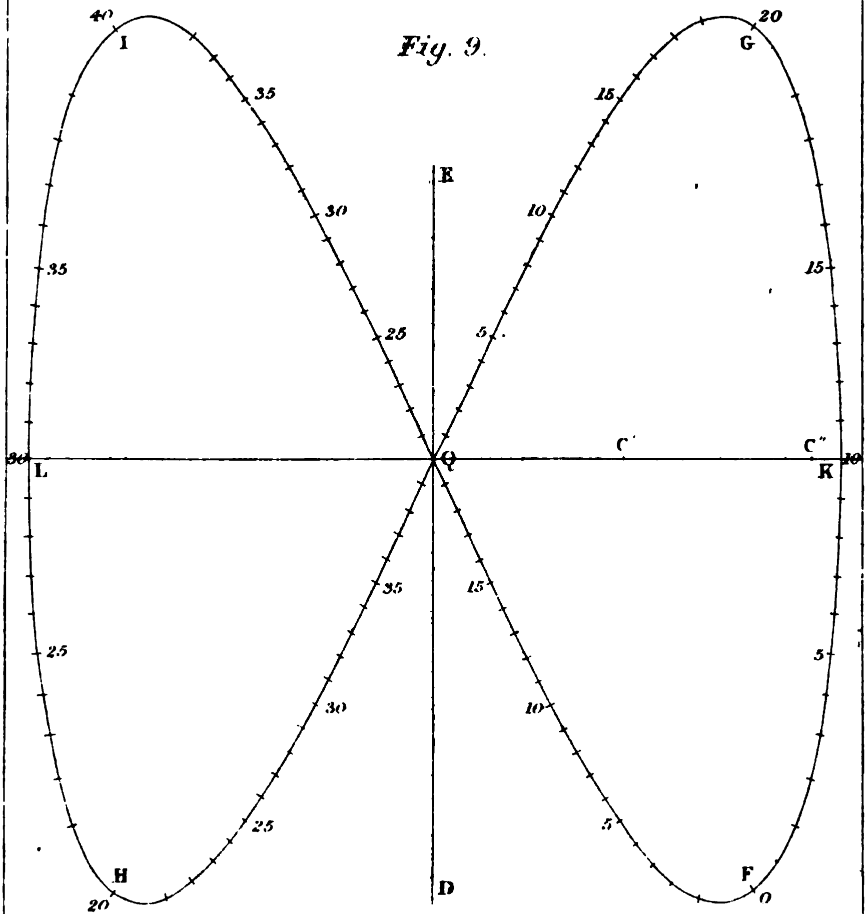

I
E
G
L
Q
C'
C''
K
H
D
F

Fig. 27.

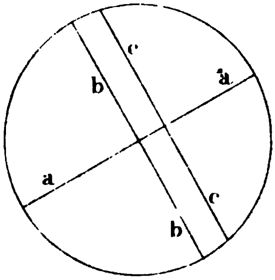

c
b
a
a
c
b

Fig. 10.

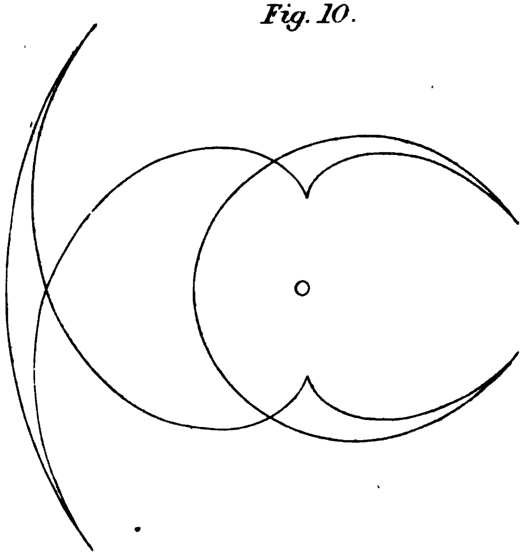


Fig. 11.

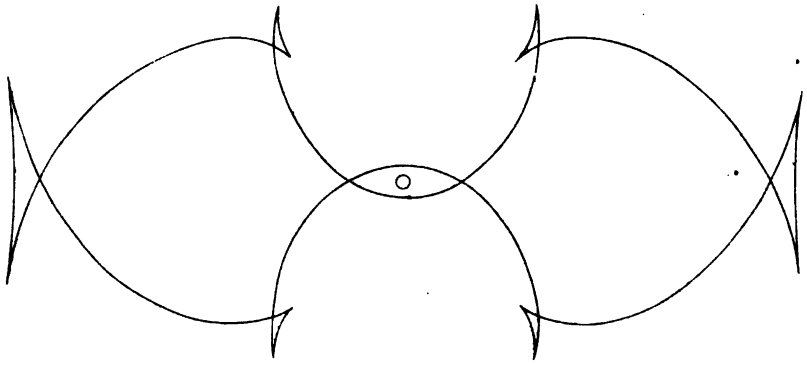


Fig. 10.

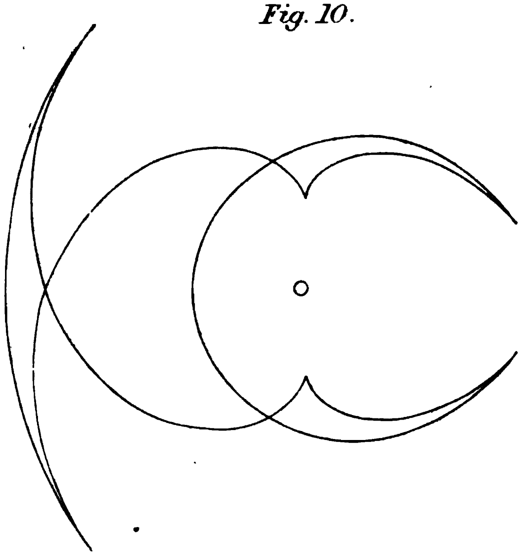


Fig. 11.

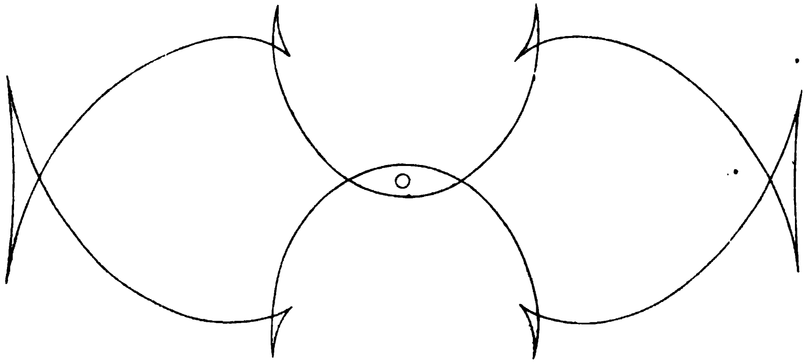


Fig. 12.

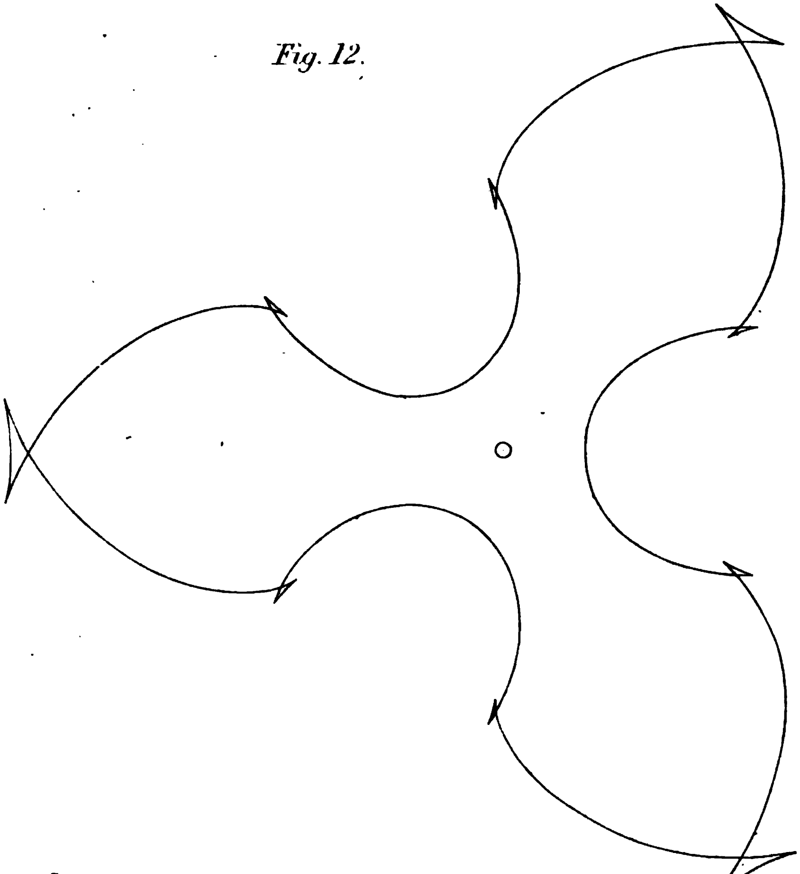
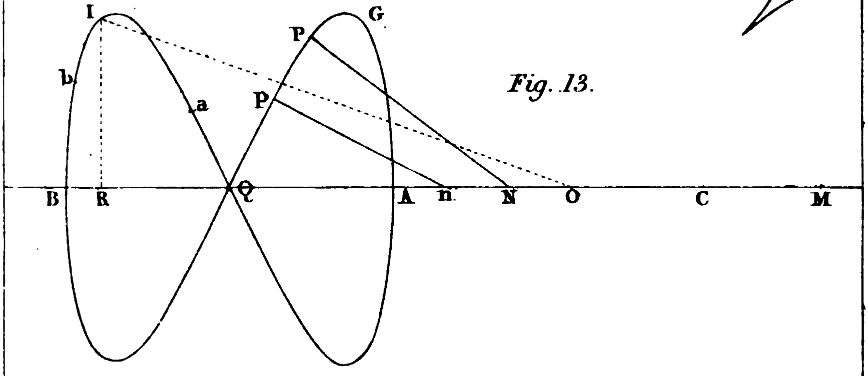


Fig. 13.



G. Aikman sc.

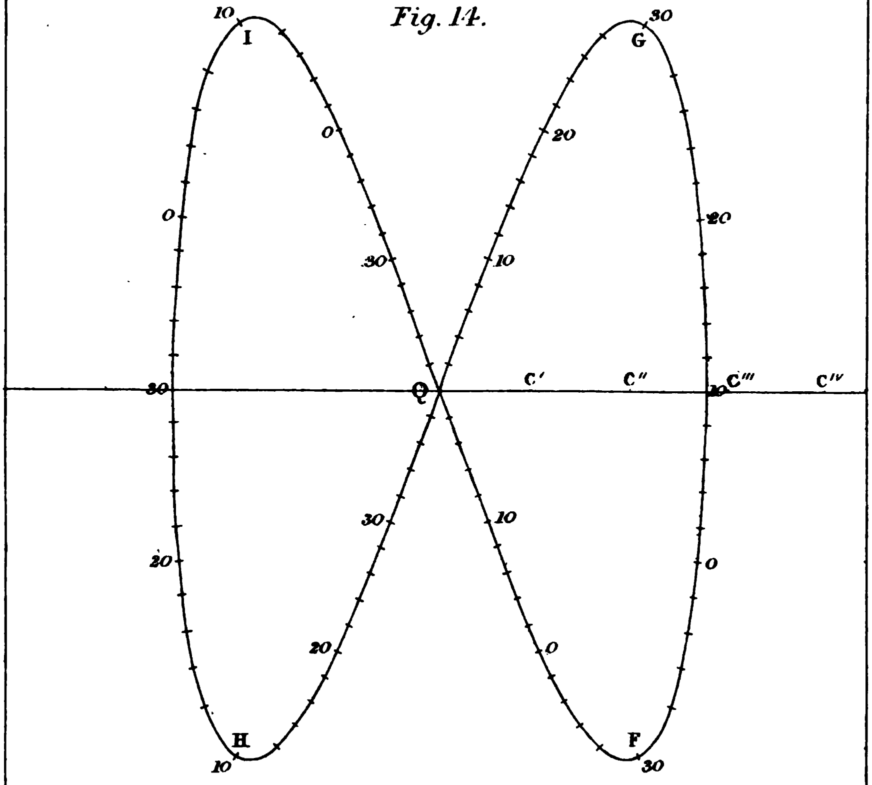


Fig. 14.

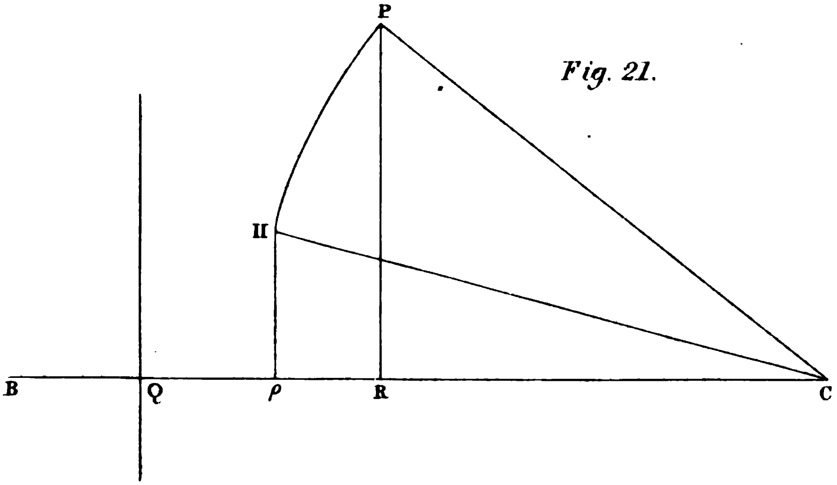


Fig. 21.

G. Aikman sc.

Fig. 15.

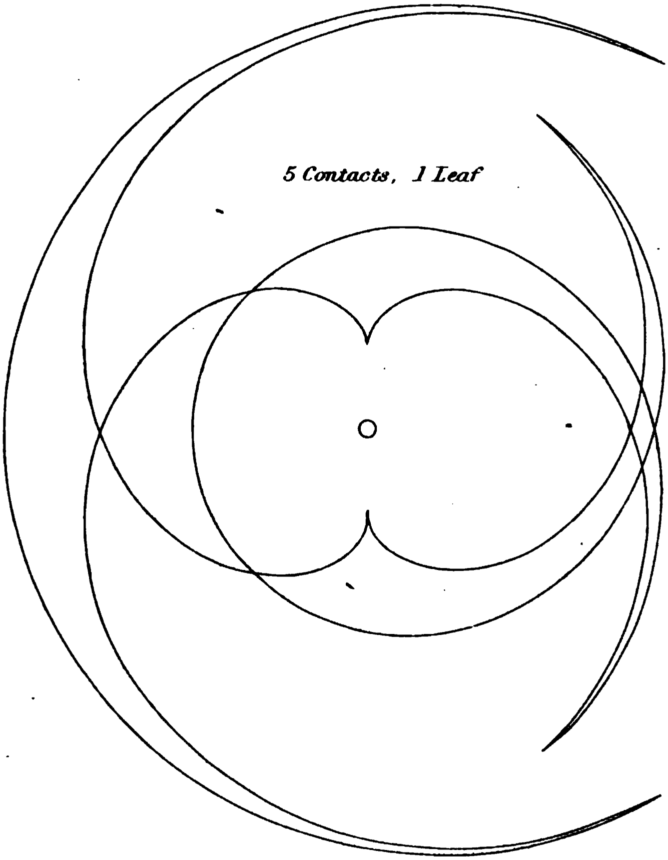
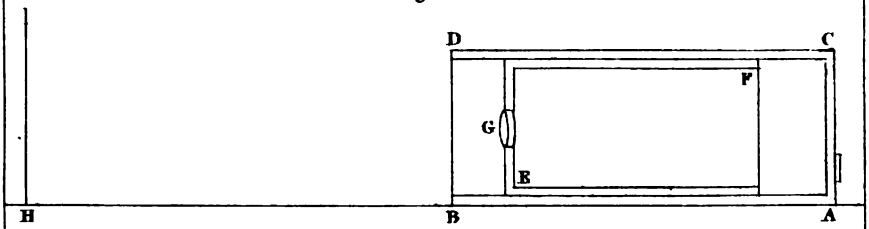


Fig. 22.



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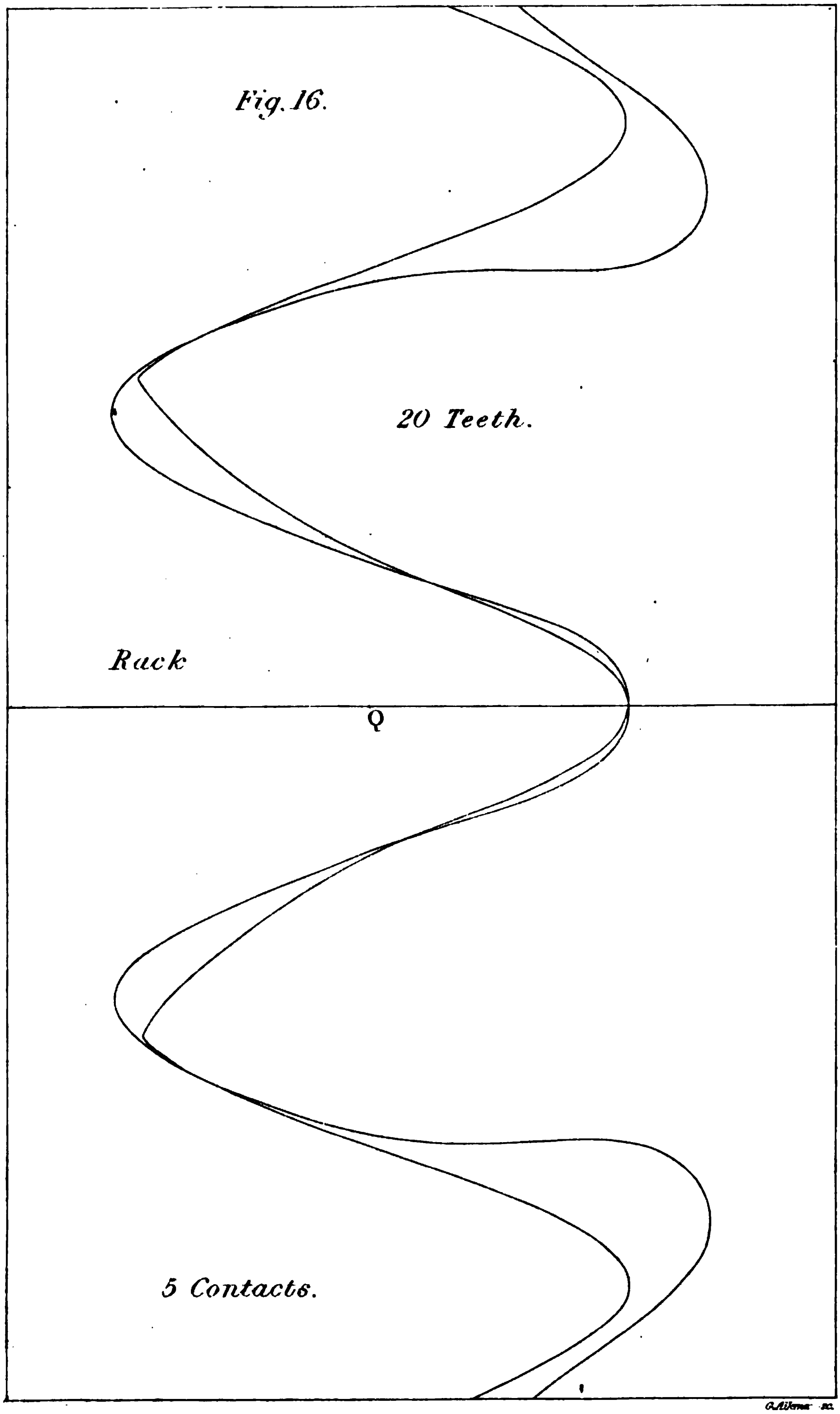
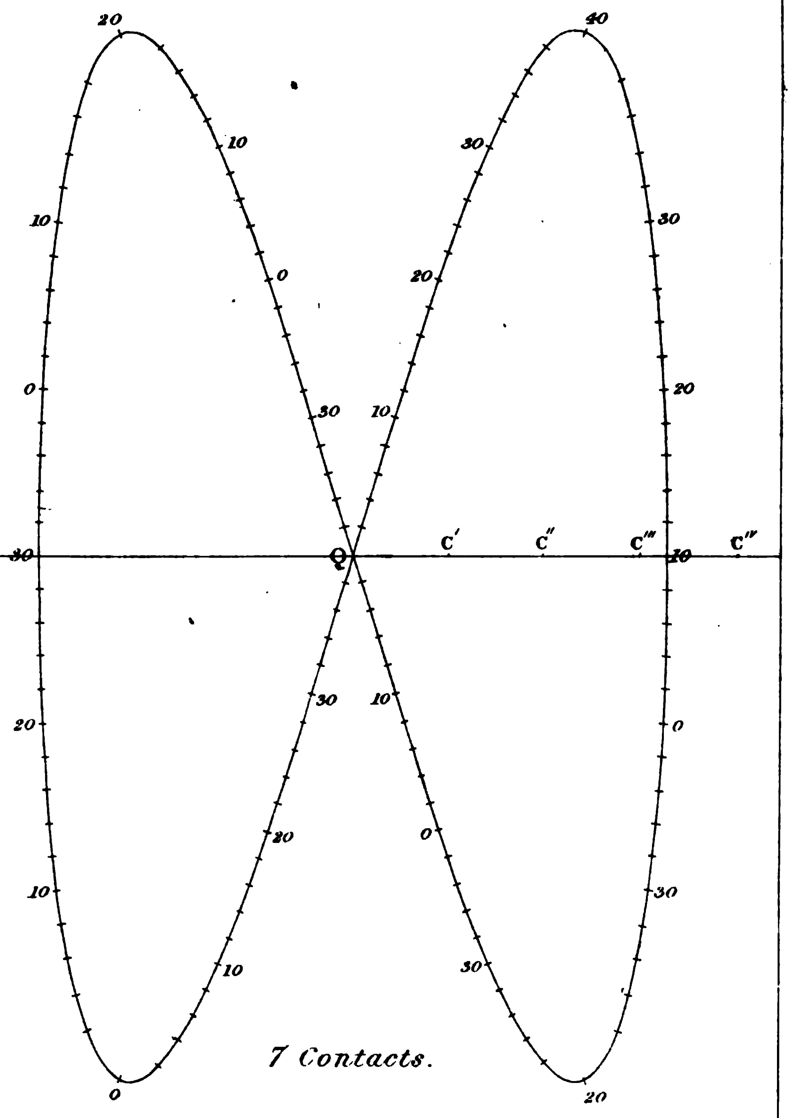


Fig. 16.

G. Aikman sc.

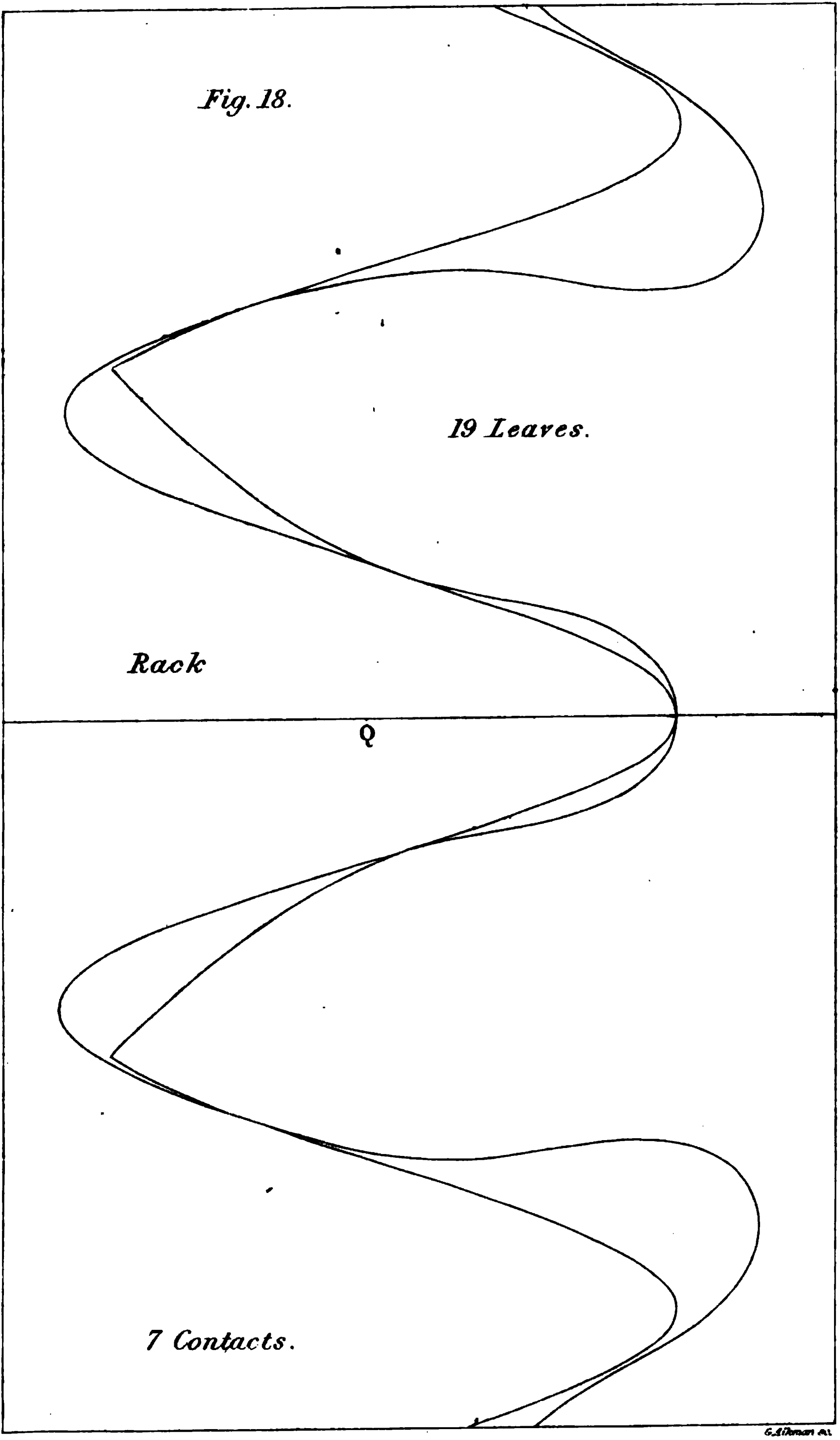
Fig. 17.



7 Contacts.

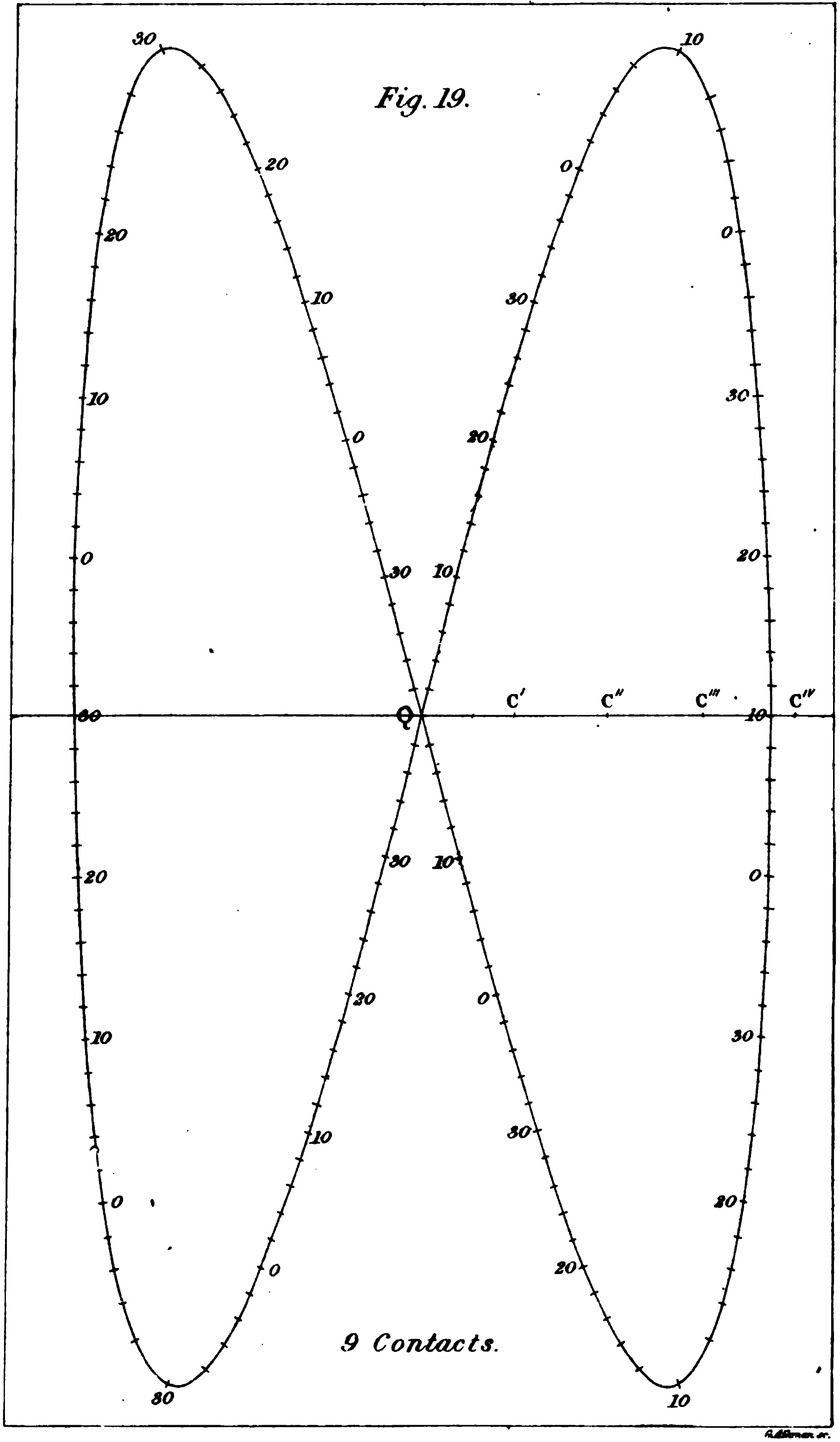
G. Allman sc.

Fig. 18.



G. Jarman sc.

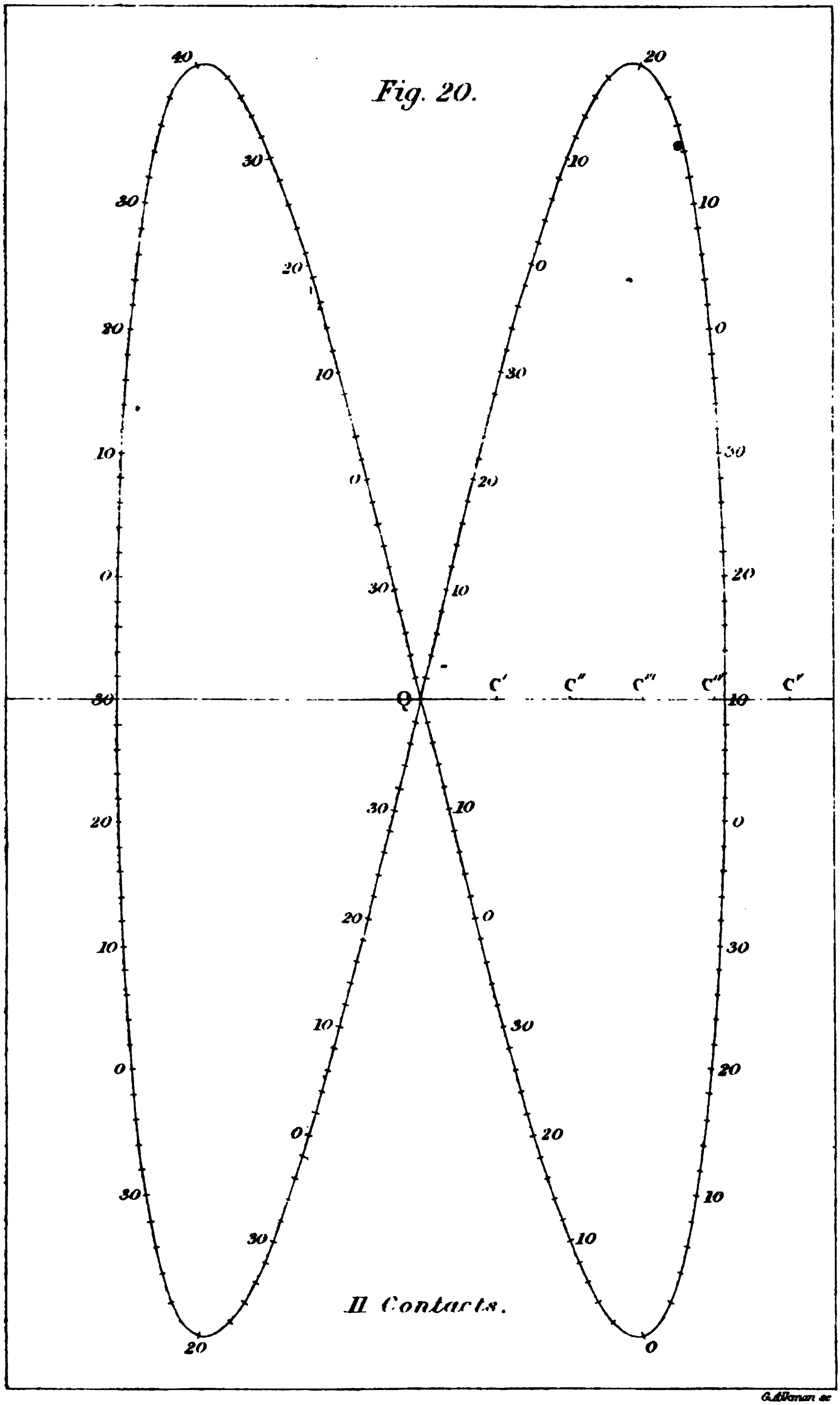
Fig. 19.



9 Contacts.

[illegible] sc.

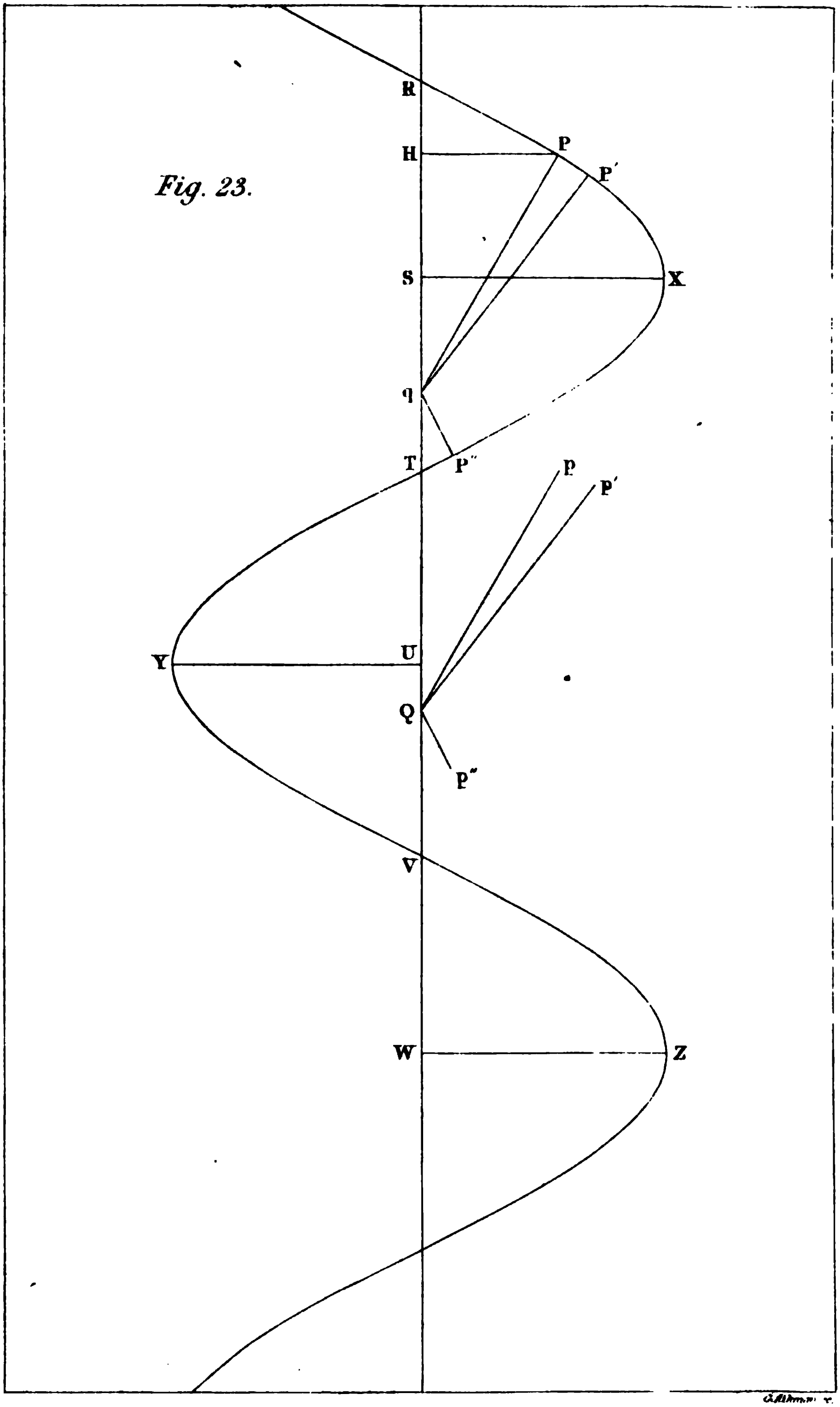
Fig. 20.

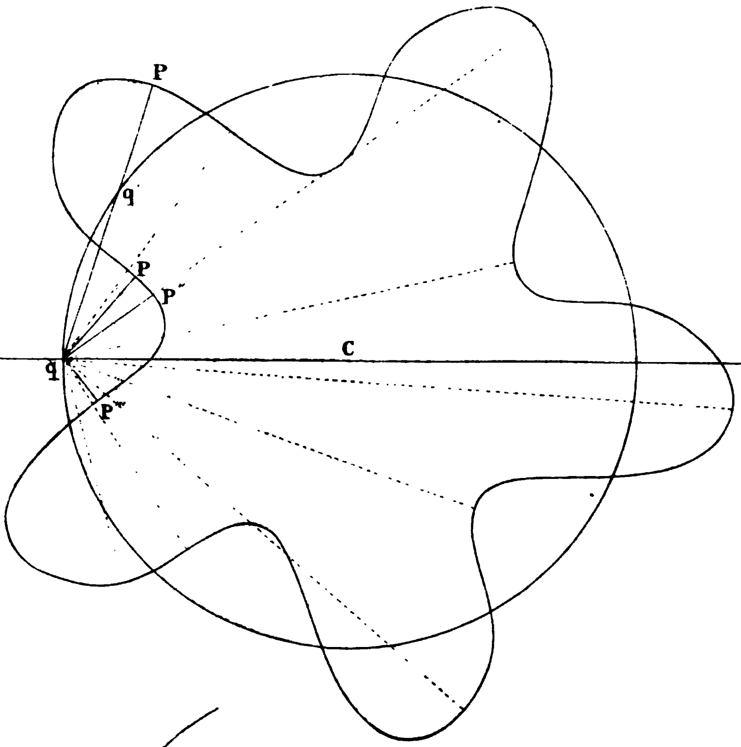


II Contacts.

G. Allman sc

Fig. 23.





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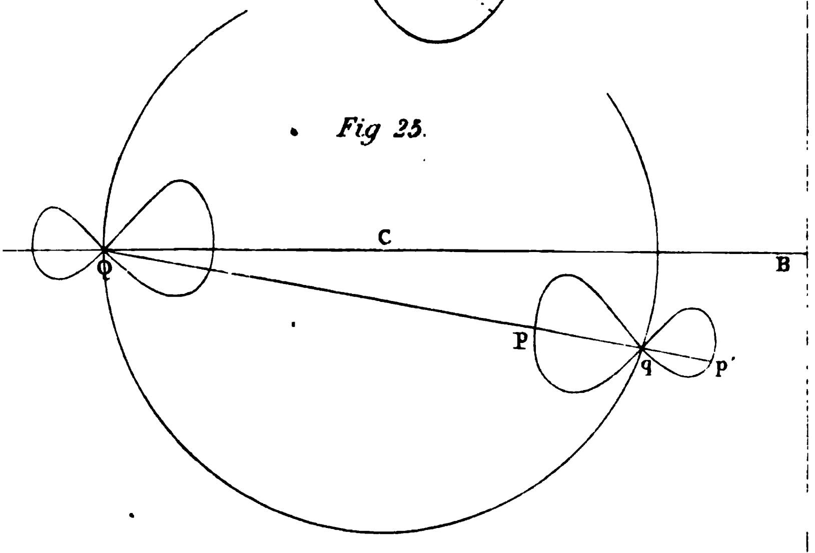


Fig 25.

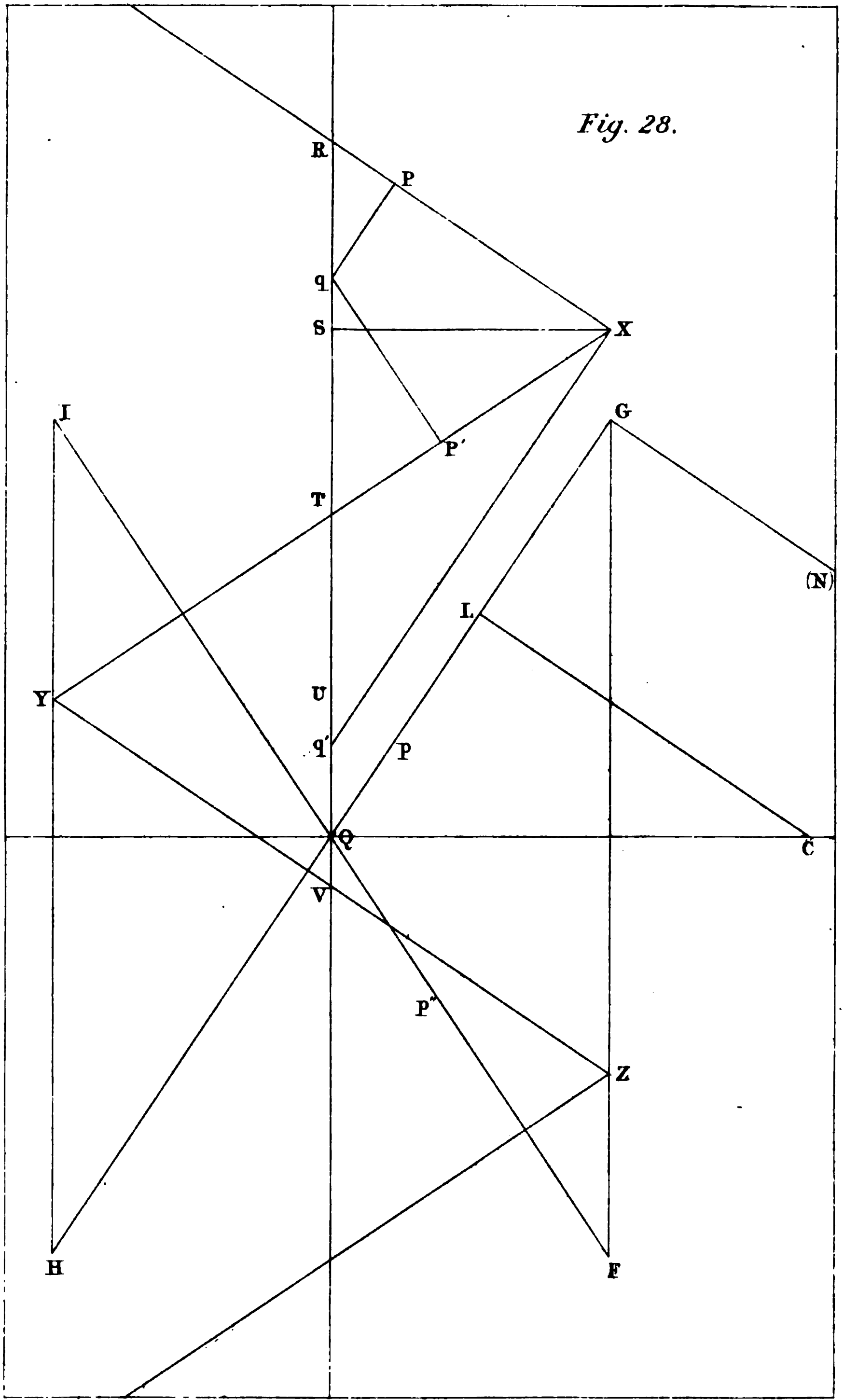
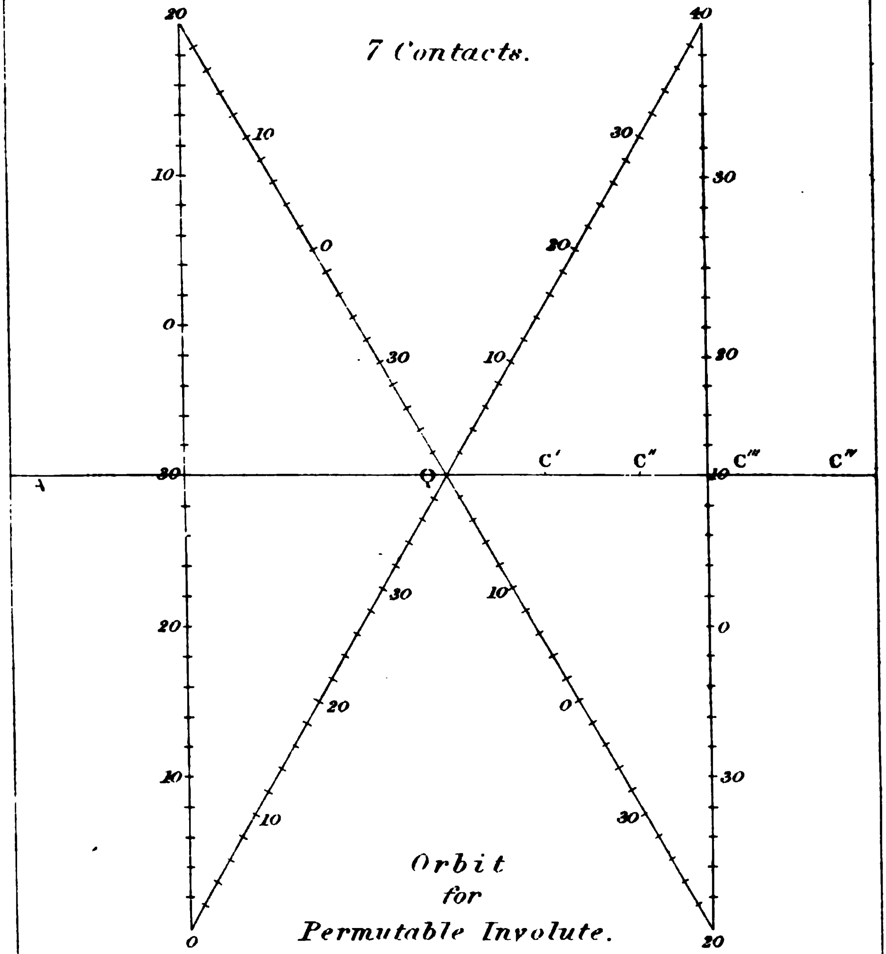


Fig. 28.

G. Aikman sc.

Fig. 29.



7 Contacts.

Orbit for Permutable Involute.

G.Aikman sc.

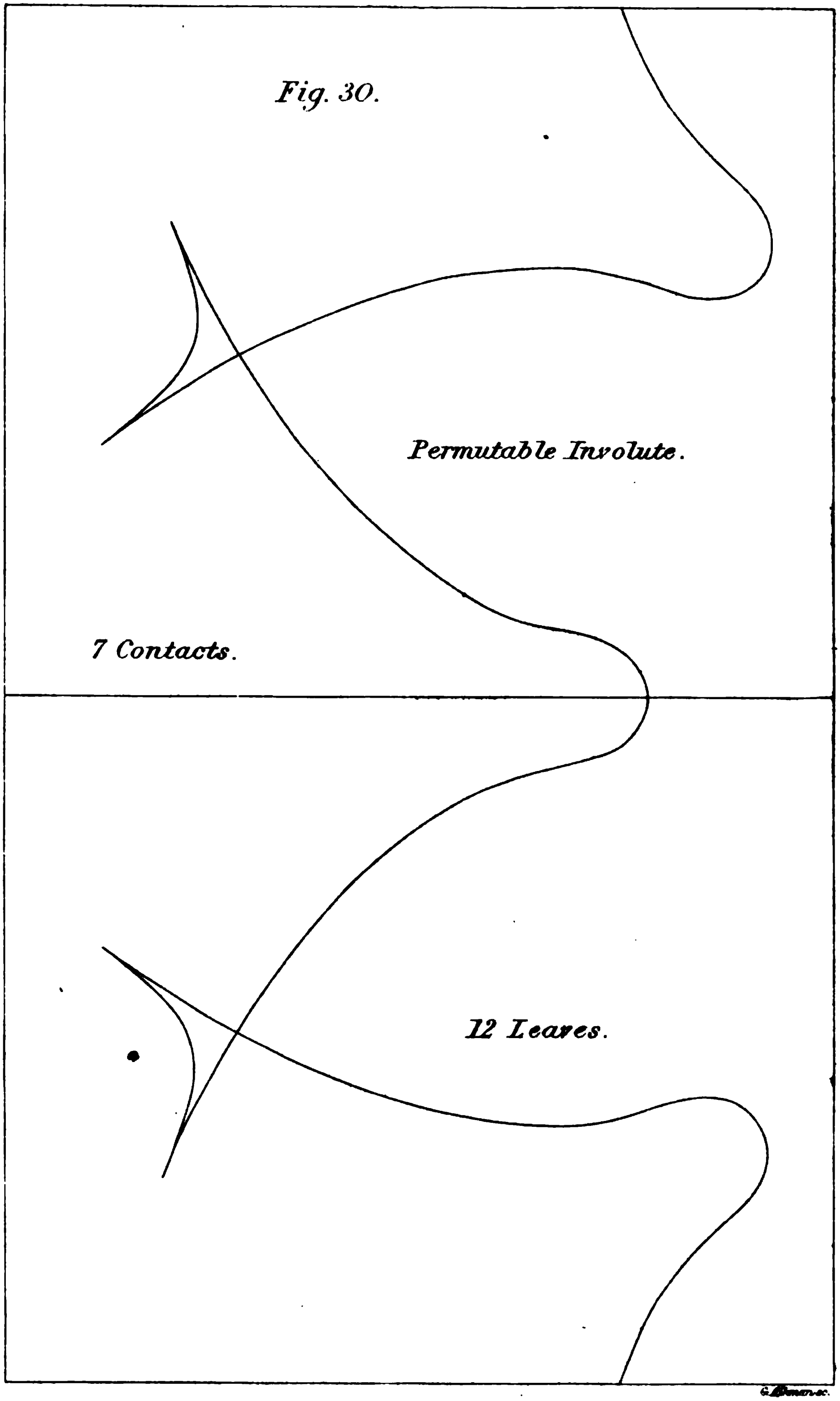


Fig. 30.

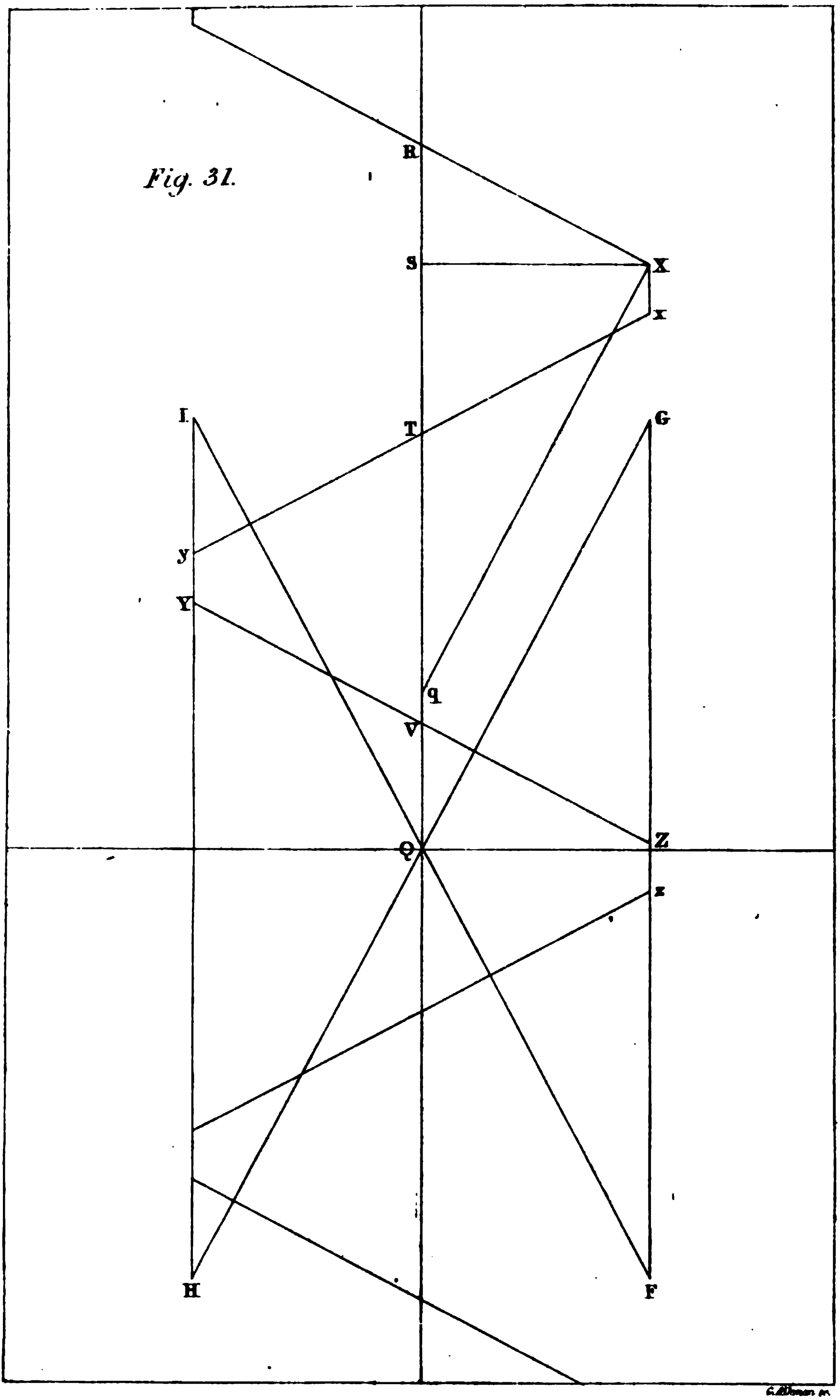
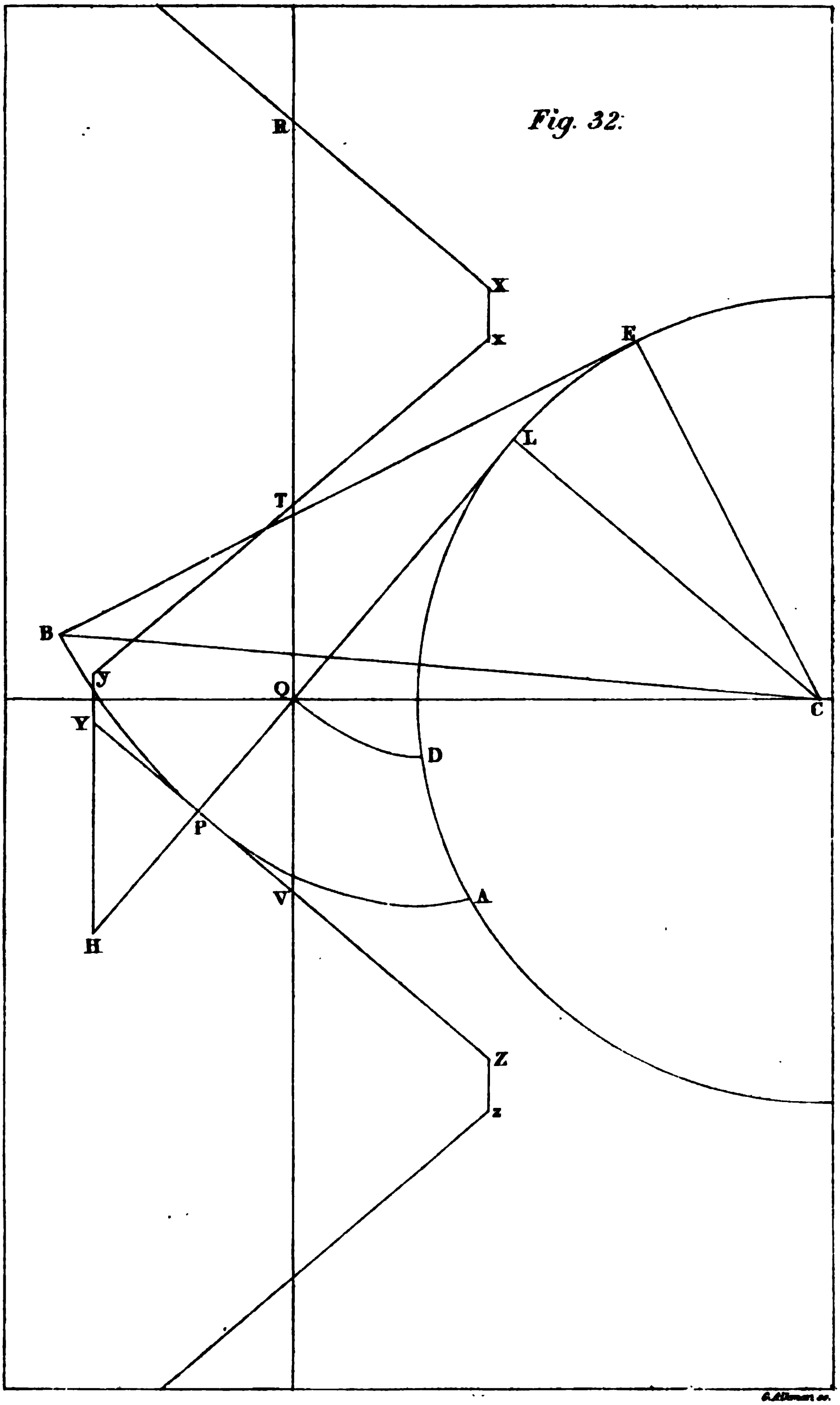


Fig. 31.

[illegible] sc.

Fig. 32.



G.Allman sc.

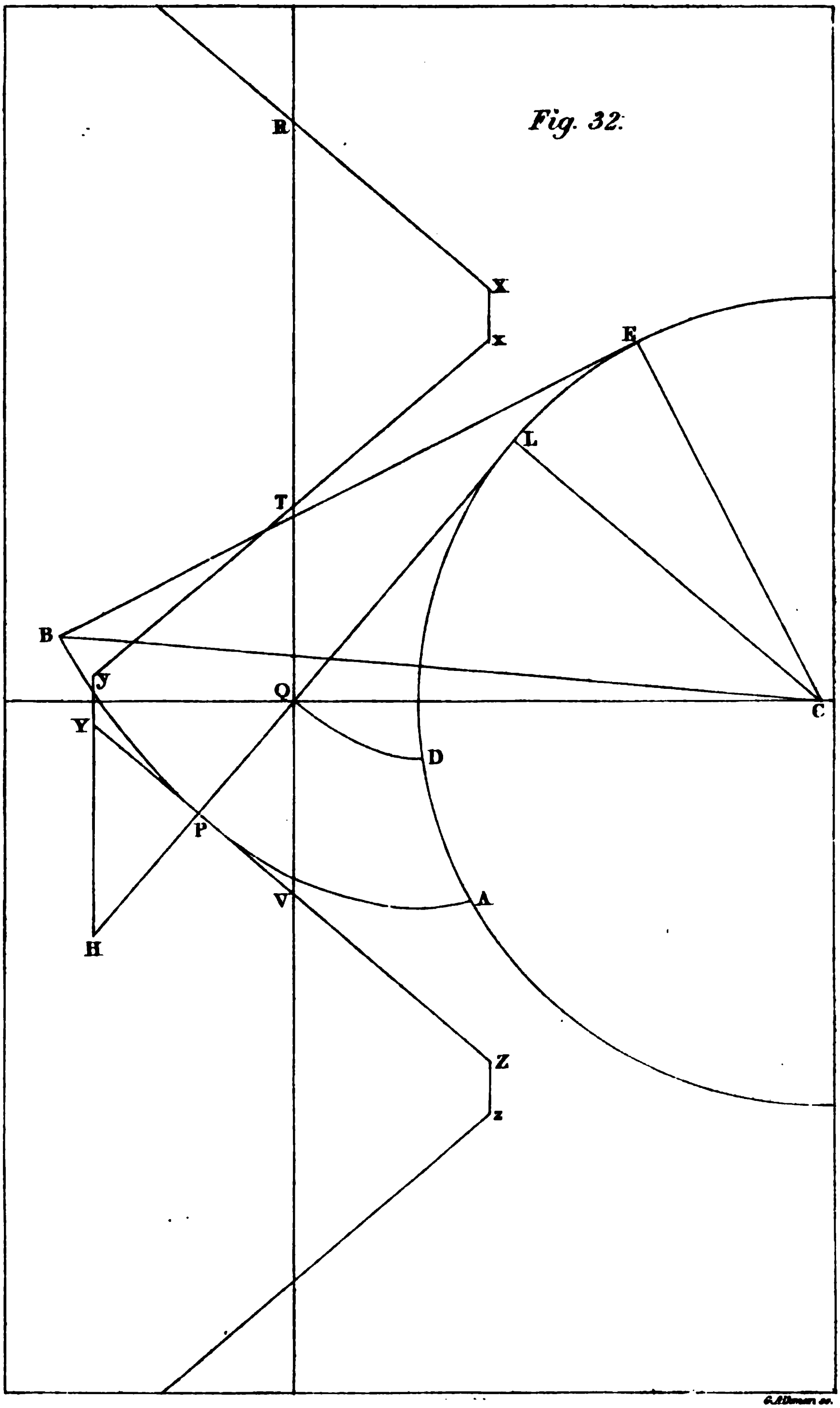


Fig. 32.

G.Aikman sc.

Fig. 34.

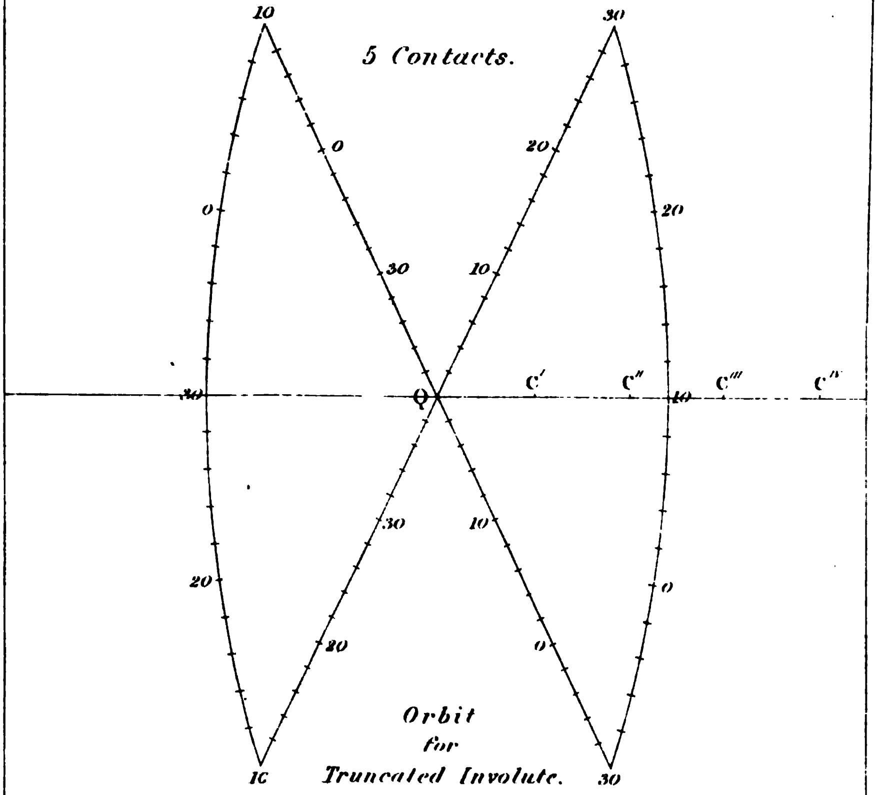
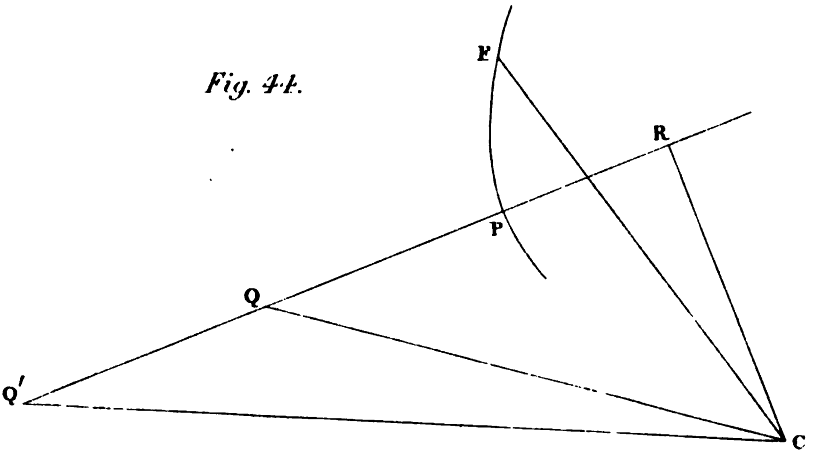


Fig. 44.



G. Aikman sc.

Fig. 34.

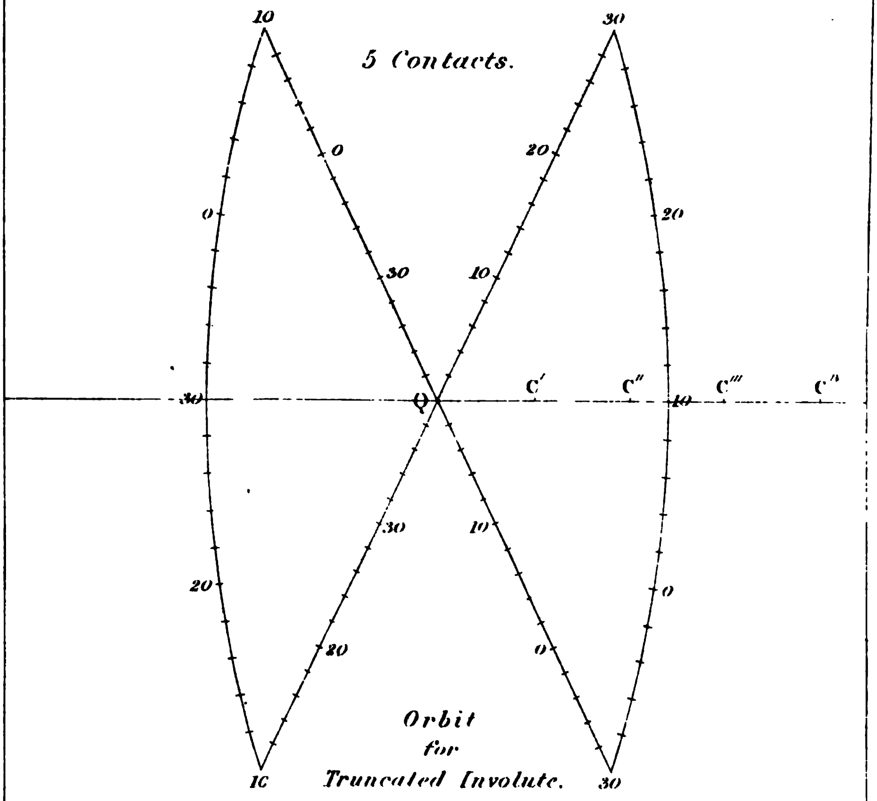
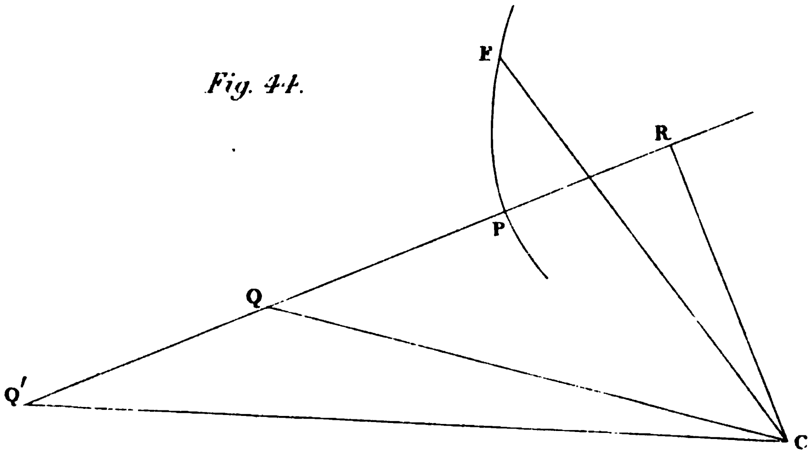


Fig. 44.



G. Aikman sc.

Fig. 35.

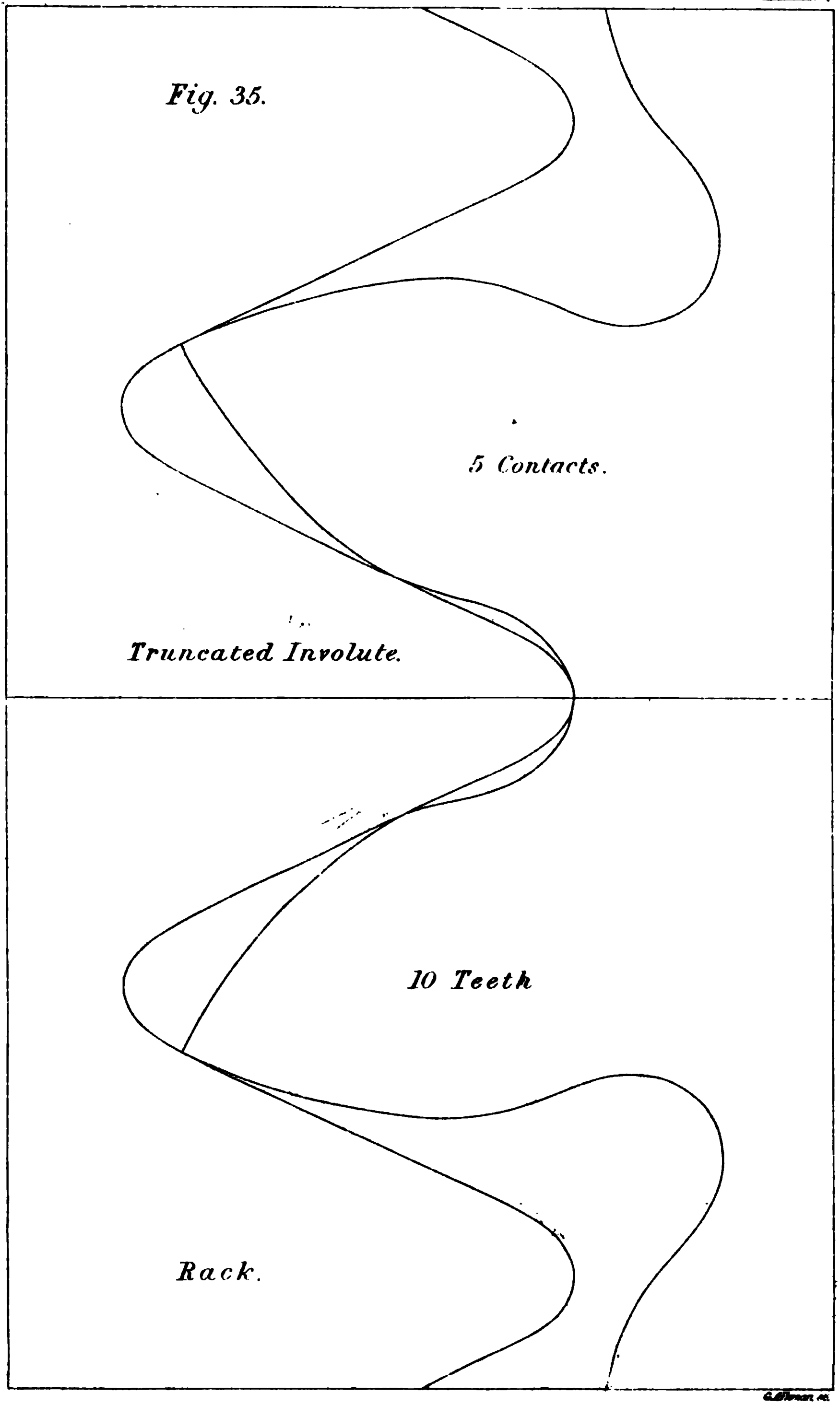
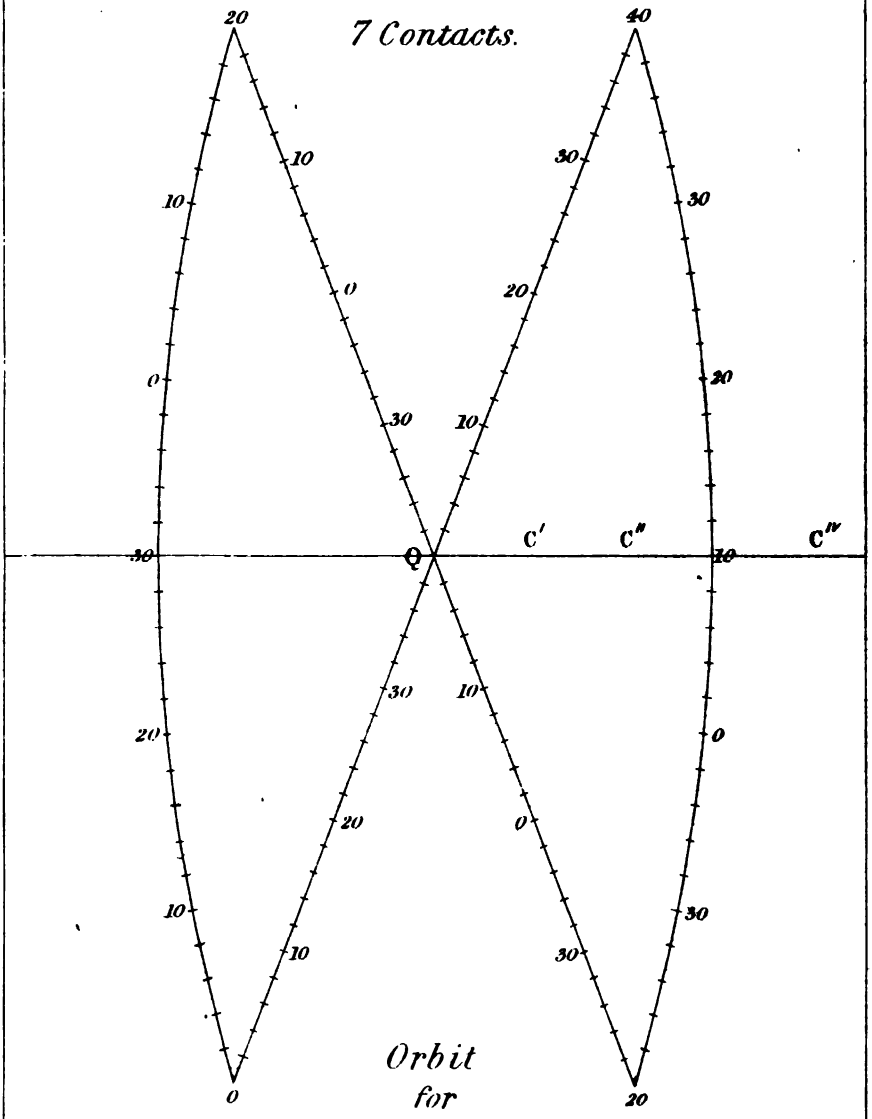


Fig. 36.

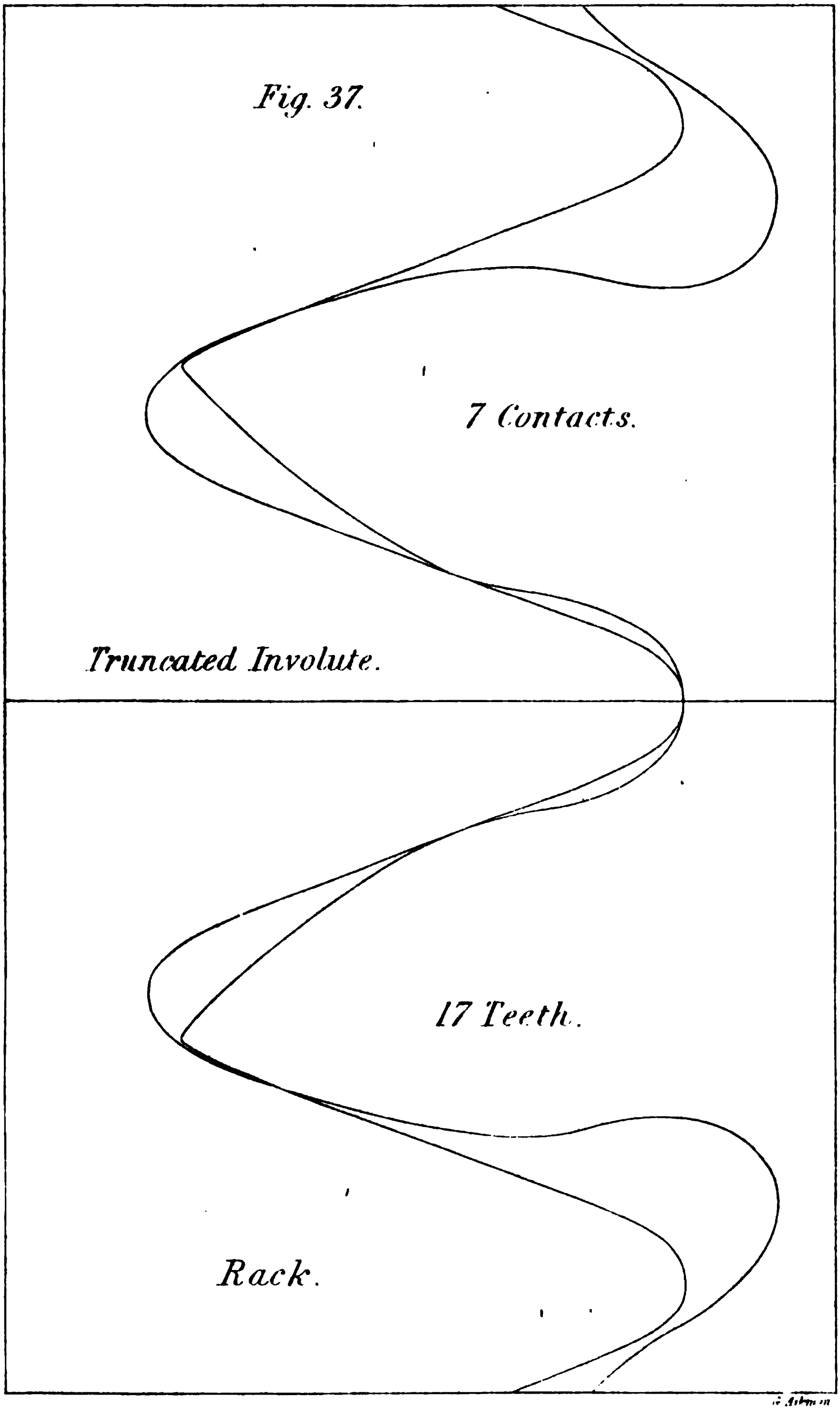
7 Contacts.



Orbit
for
Truncated Involute.

G. Aikman

Fig. 37.



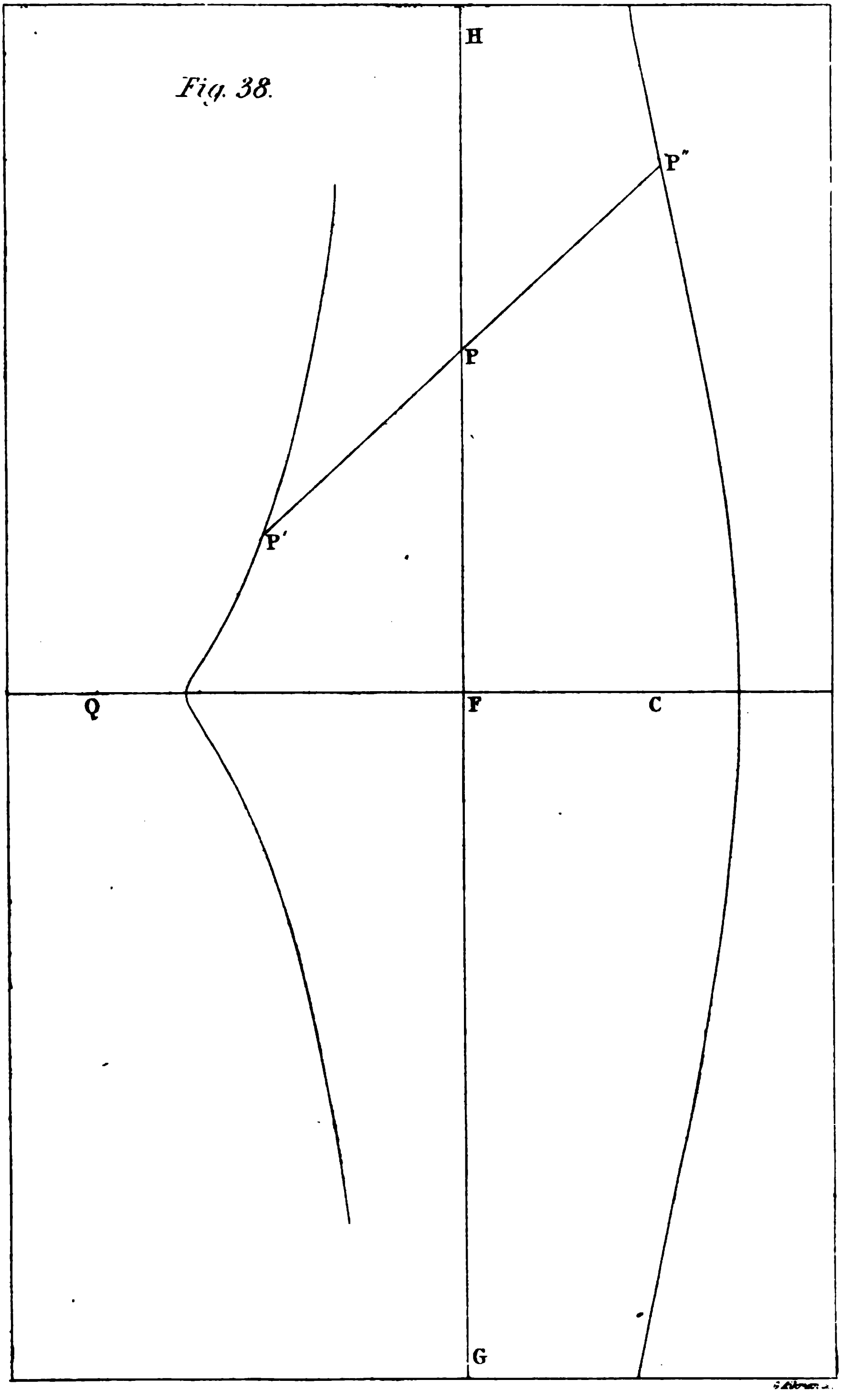
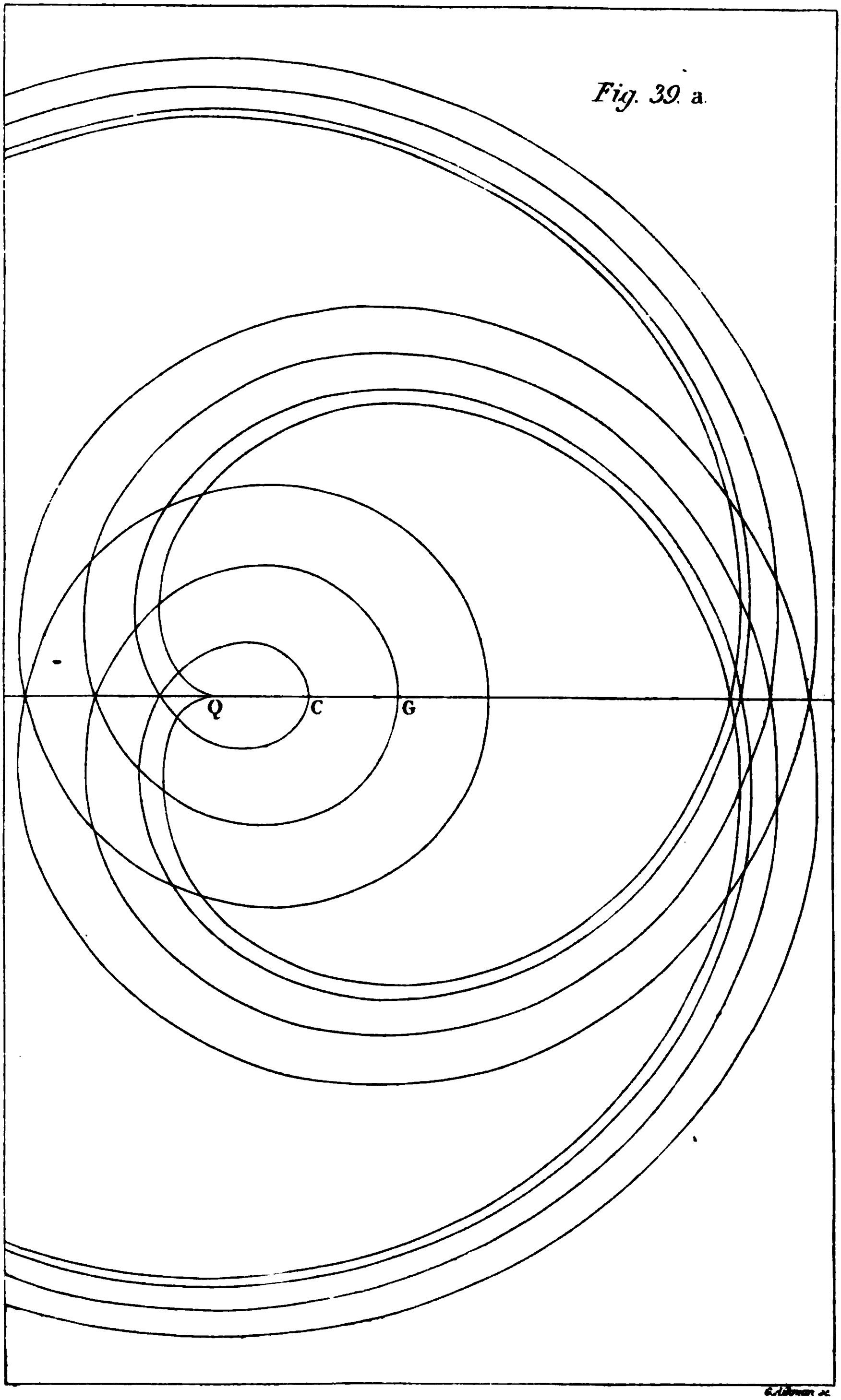


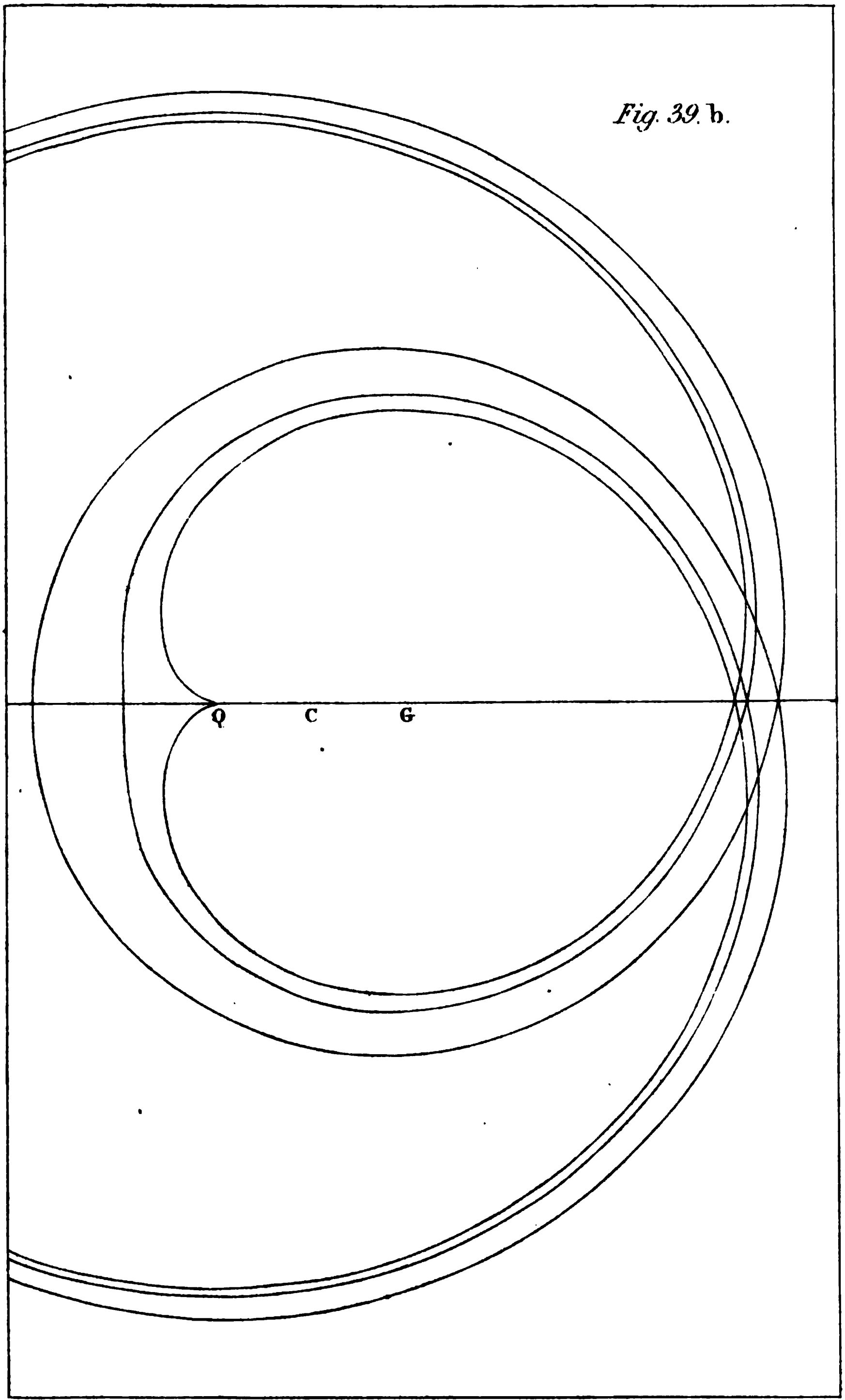
Fig. 38.
H
P″
P
P′
Q
F
C
G

Fig. 39. a.



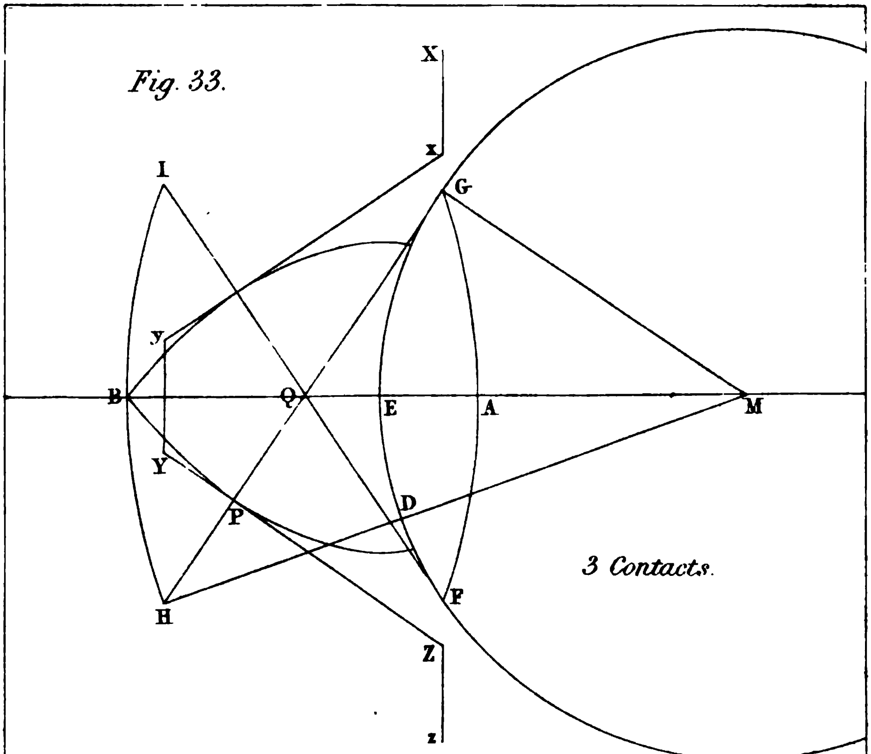
G. Aikman sc.

Fig. 39. b.



G. Aikman sc.

Fig. 33.



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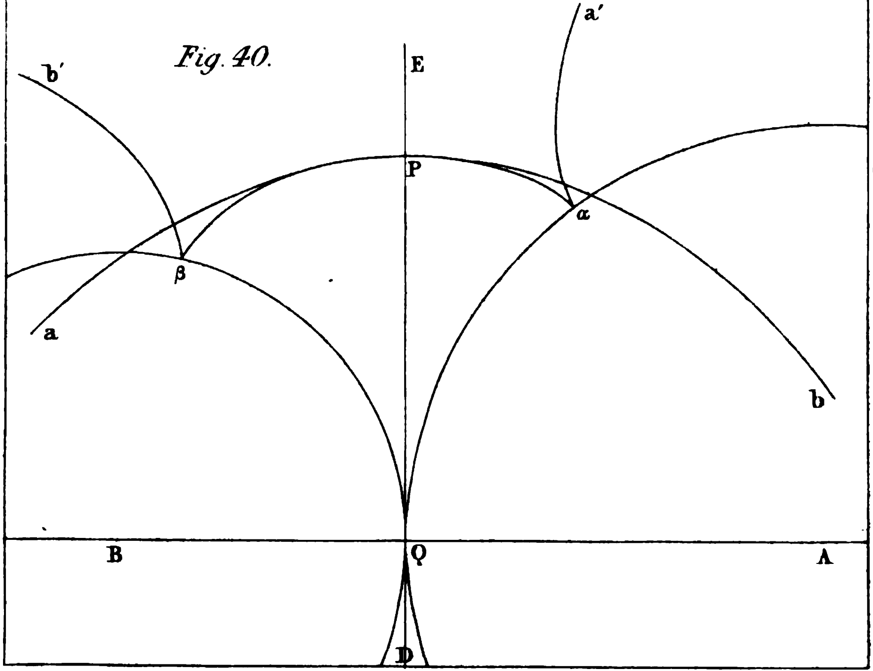


G. Aikman sc.

Fig. 41.

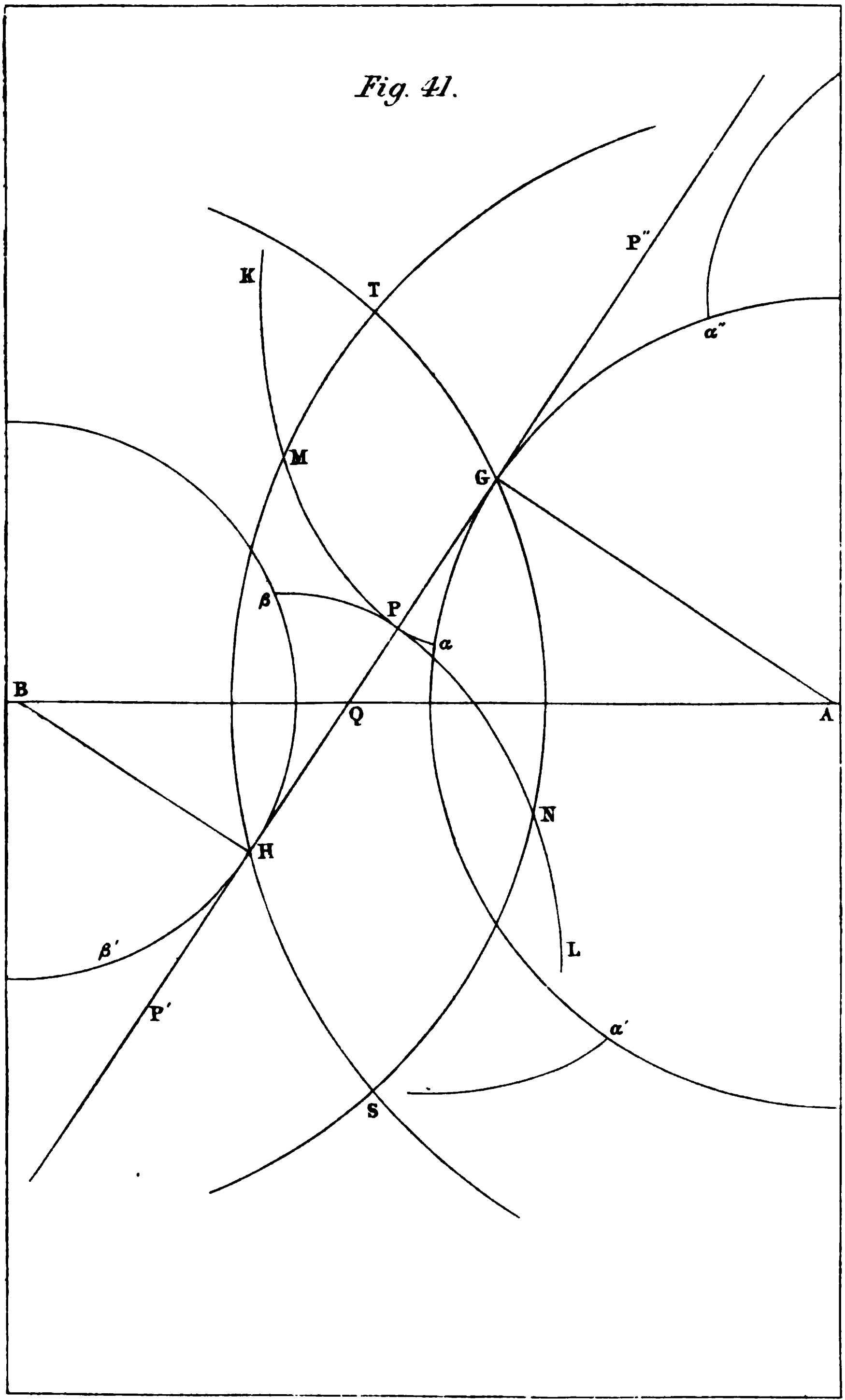
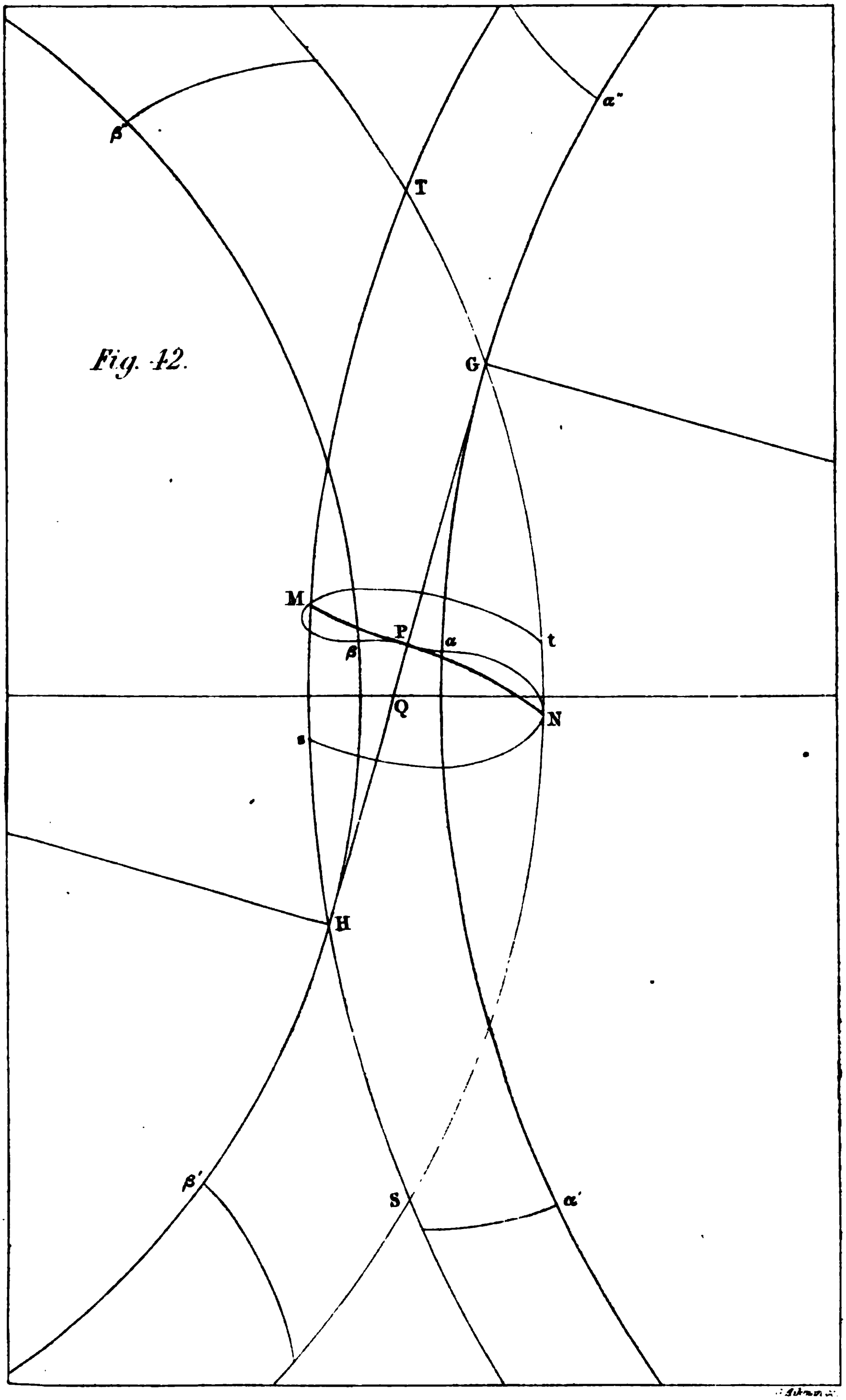


Fig. 12.



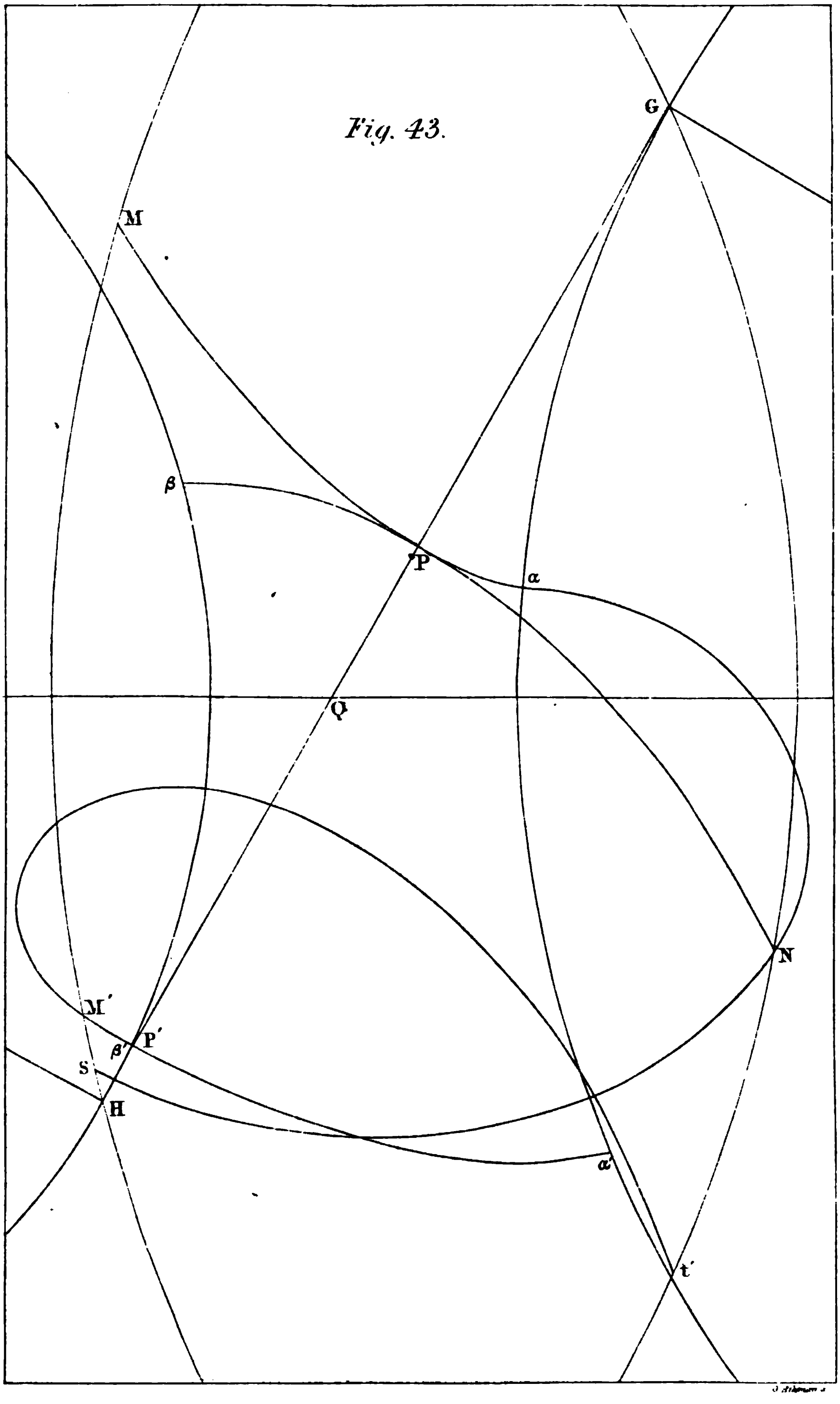


Fig. 43.

J. Atkinson sc.

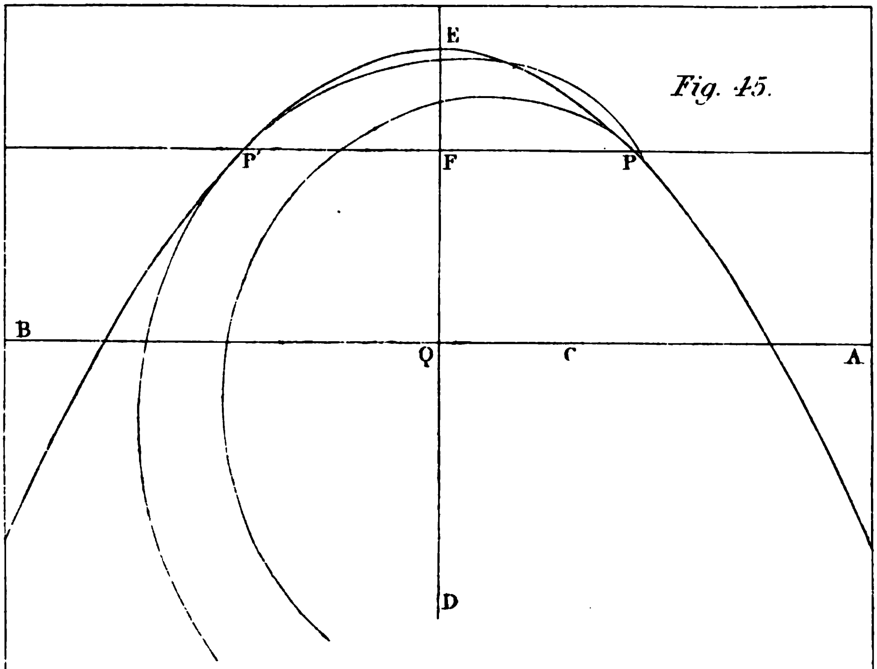
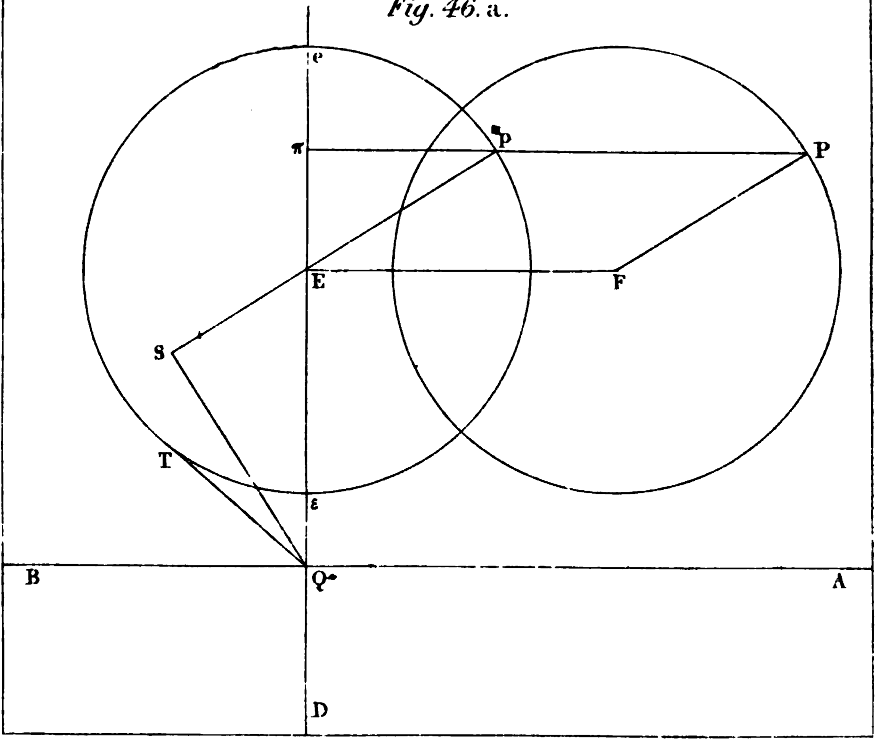


Fig. 45.

Fig. 46. a.



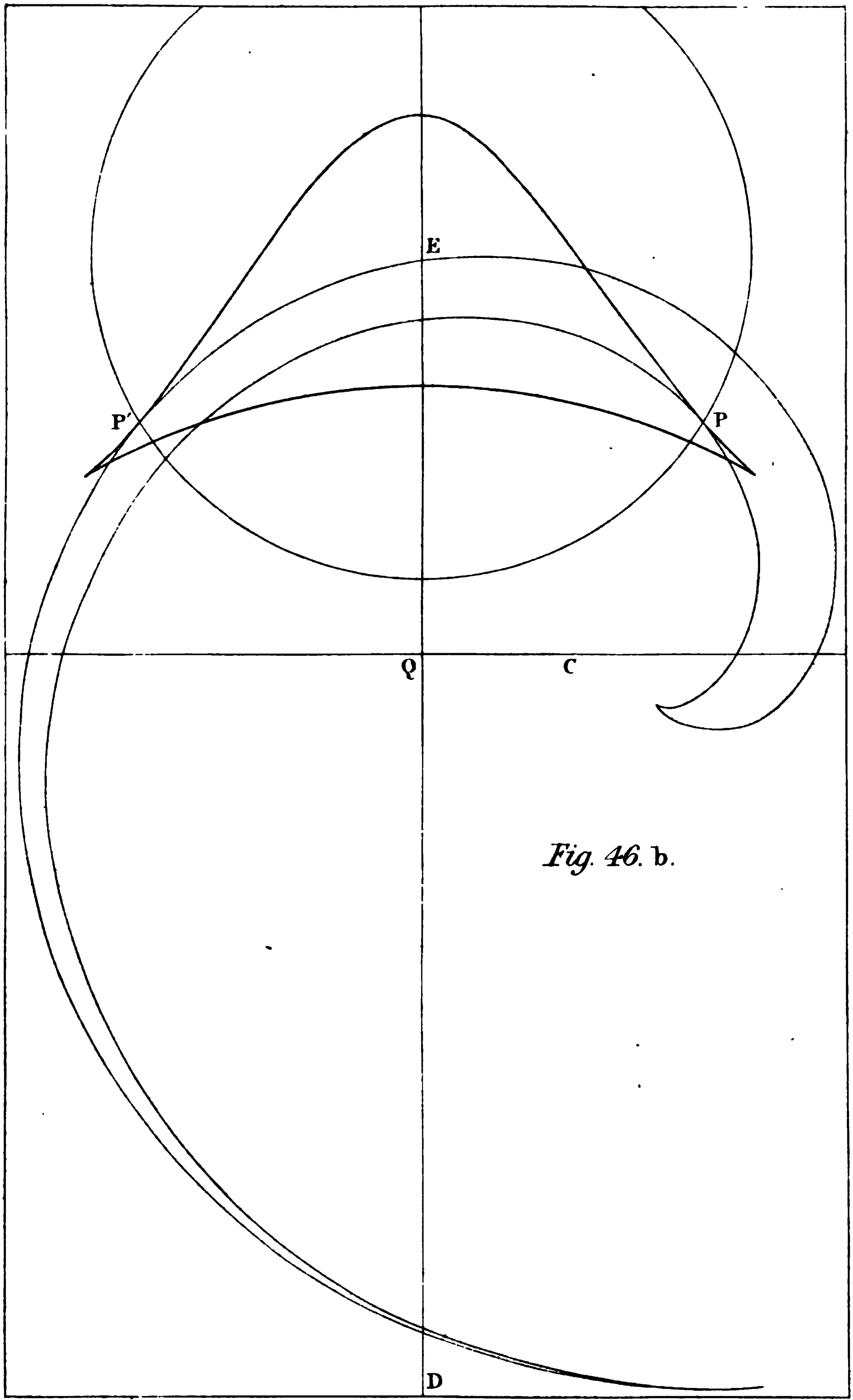


Fig. 46. b.

Fig. 48.

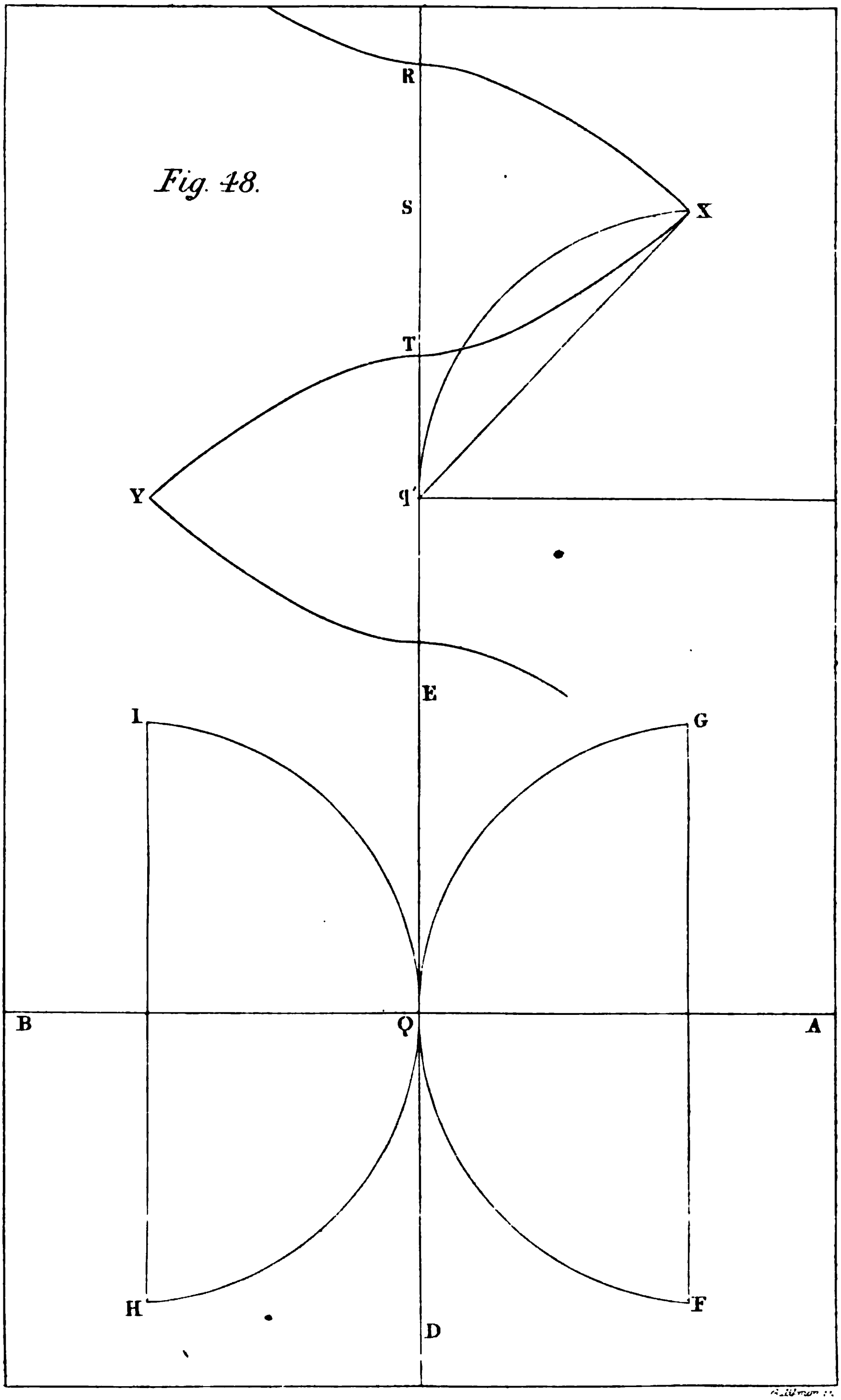


Fig. 49.

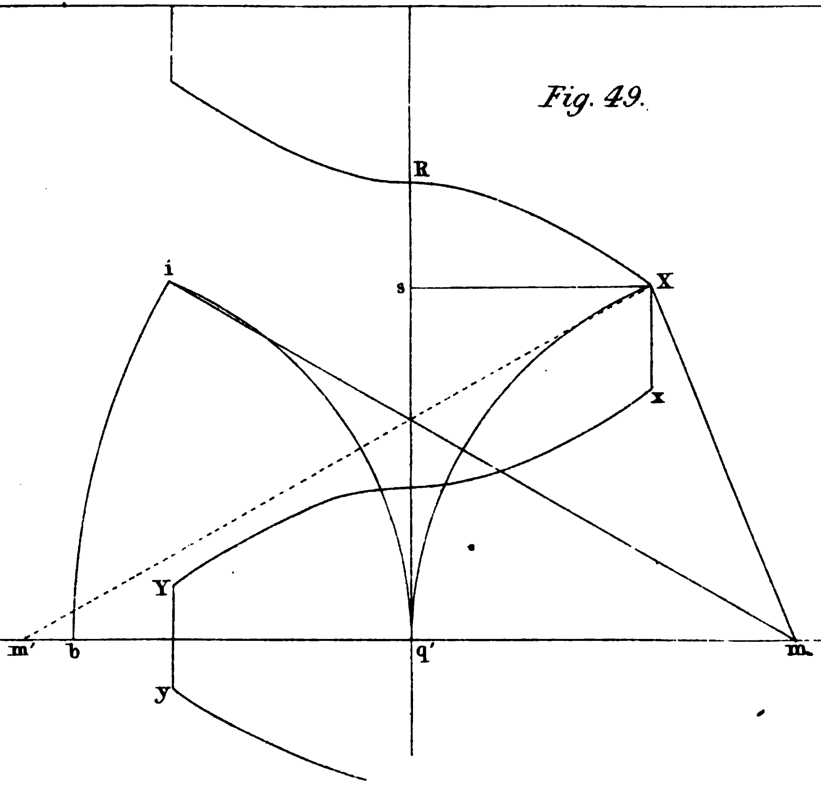
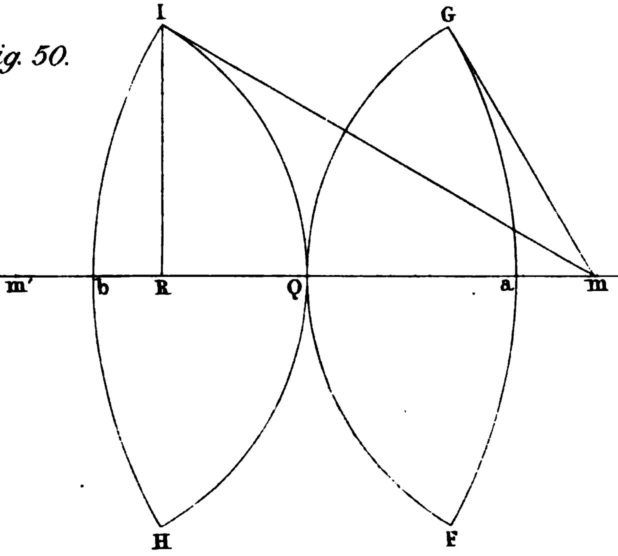


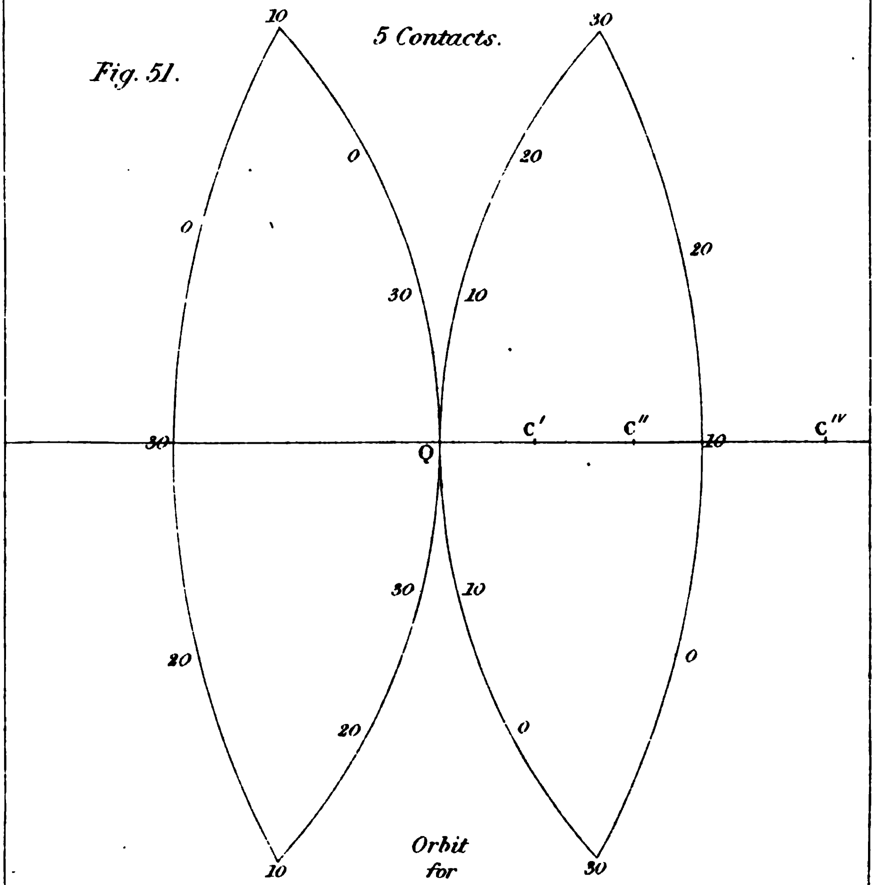
Fig. 50.



G. Aikman sc.

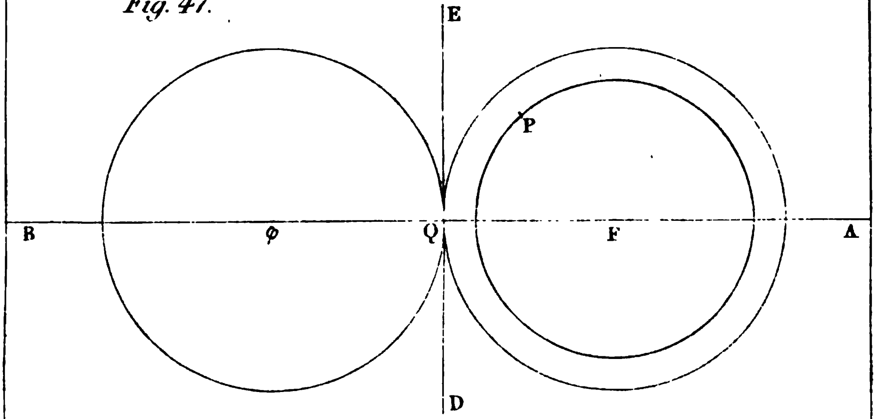
Fig. 51.

5 Contacts.



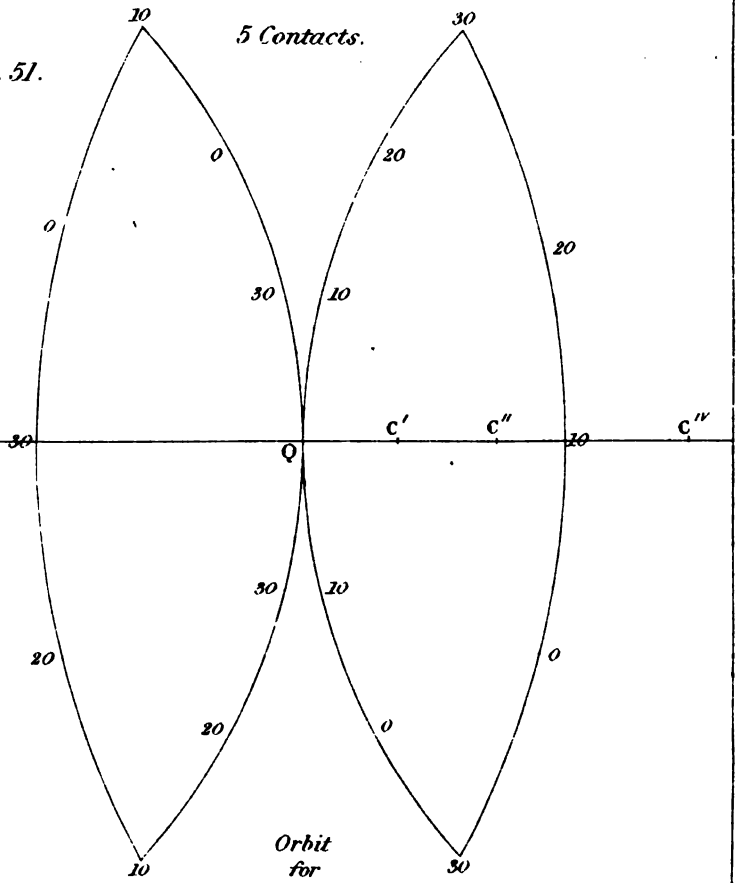
Orbit
for
Truncated Epicycloid.

Fig. 47.



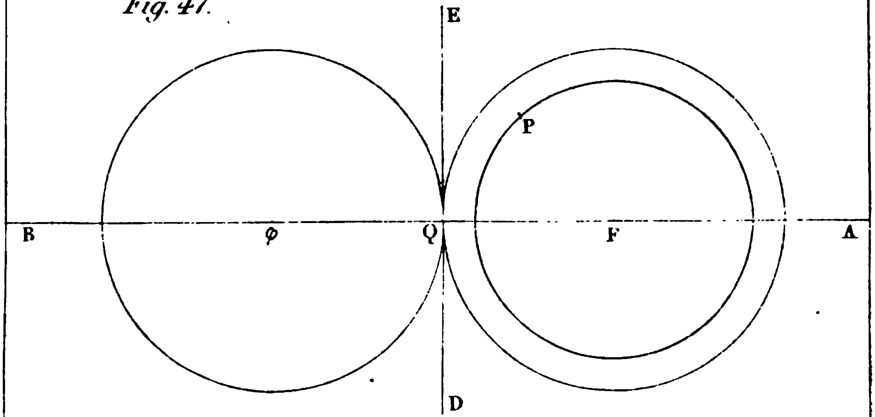
G. Aikman sc.

Fig. 51.



Orbit for Truncated Epicycloid.

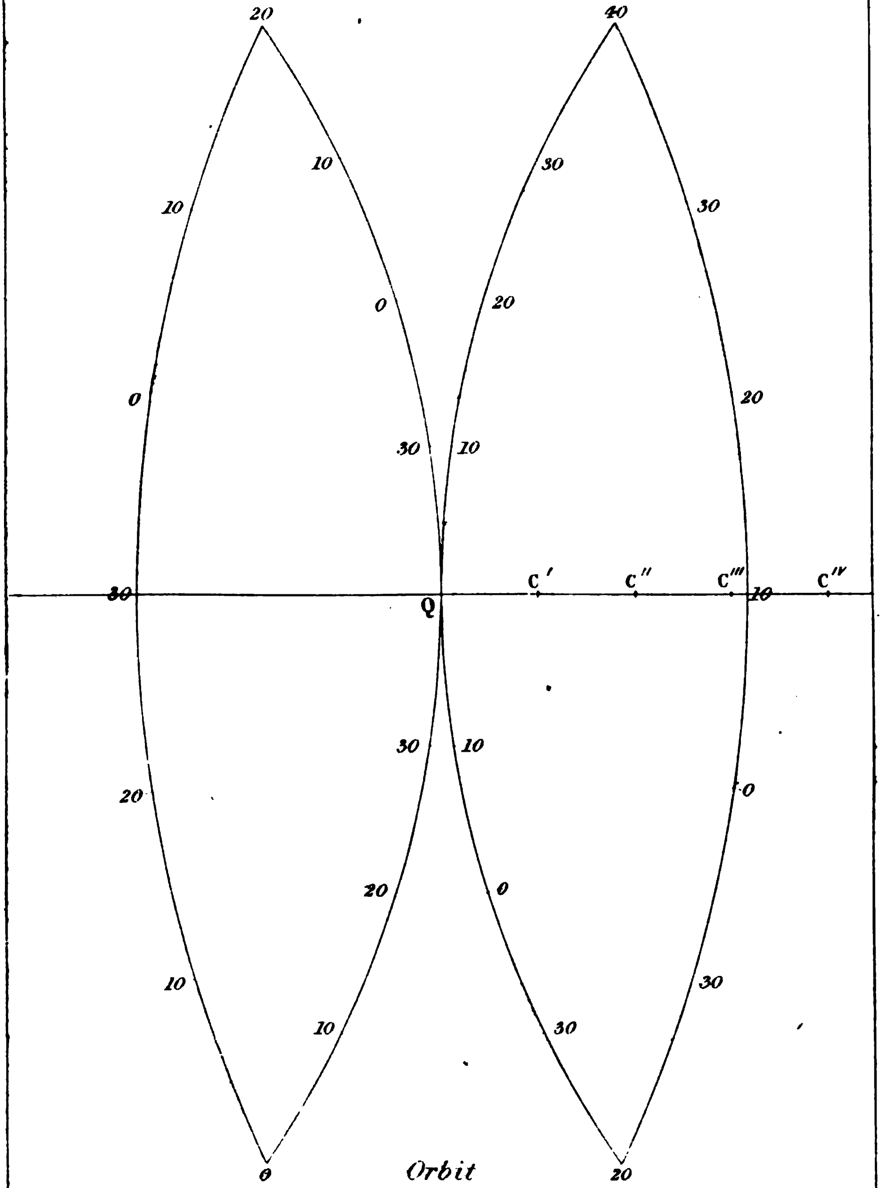
Fig. 47.



G. Aikman sc.

Fig. 52.

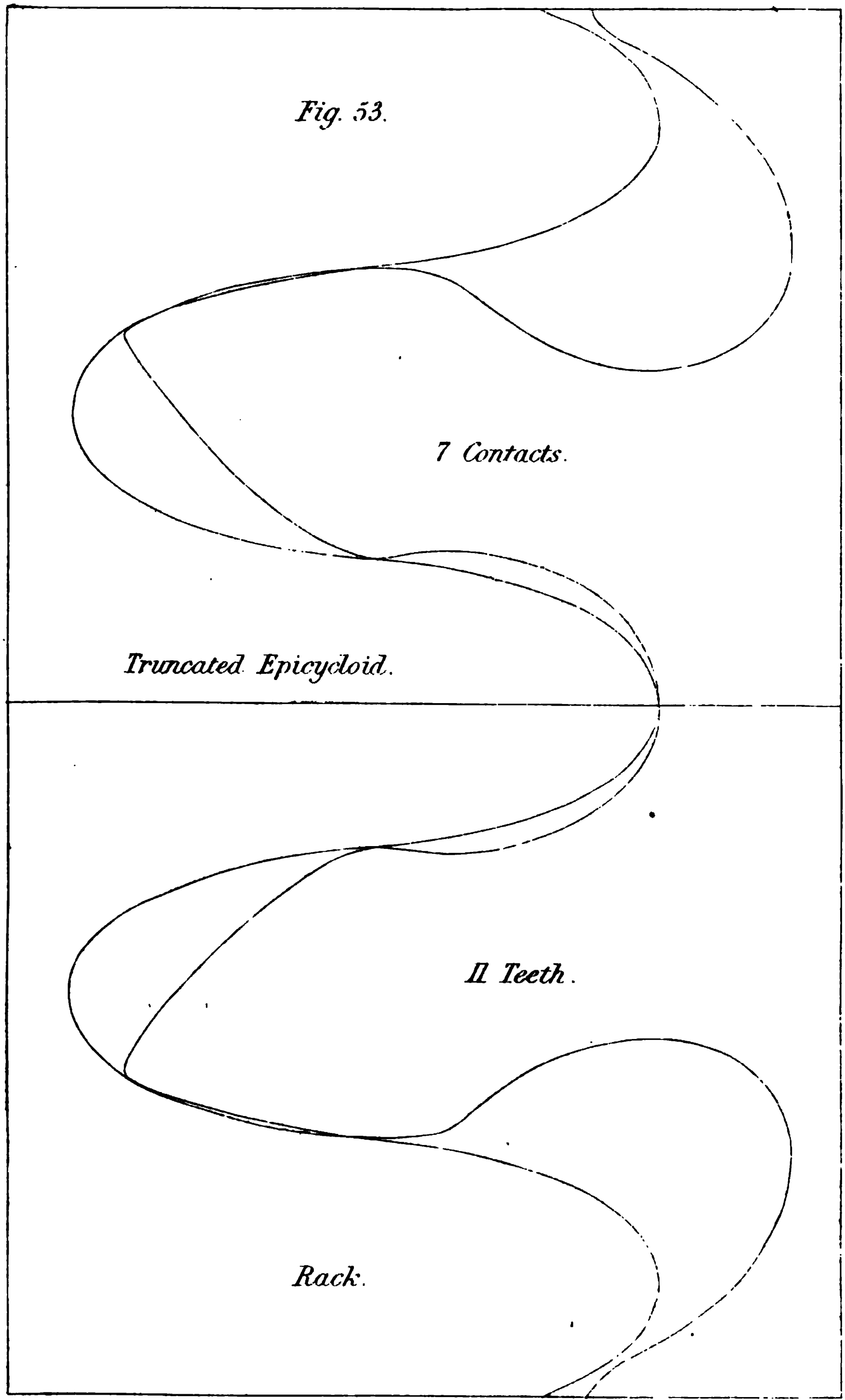
7 Contacts.



Orbit
for
Truncated Epicycloid.

G. Aikman sc.

Fig. 53.



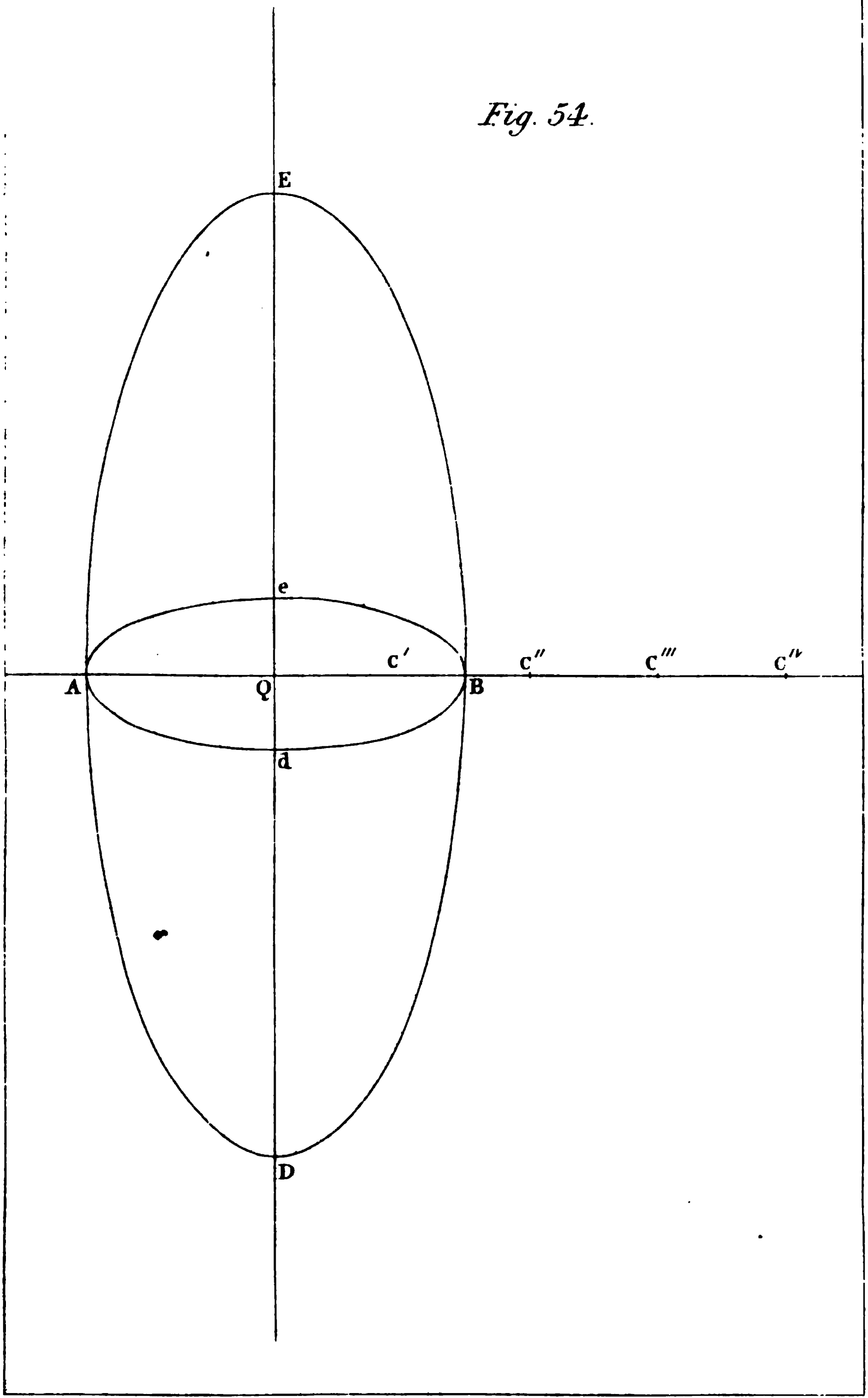


Fig. 54.

Fig. 55.

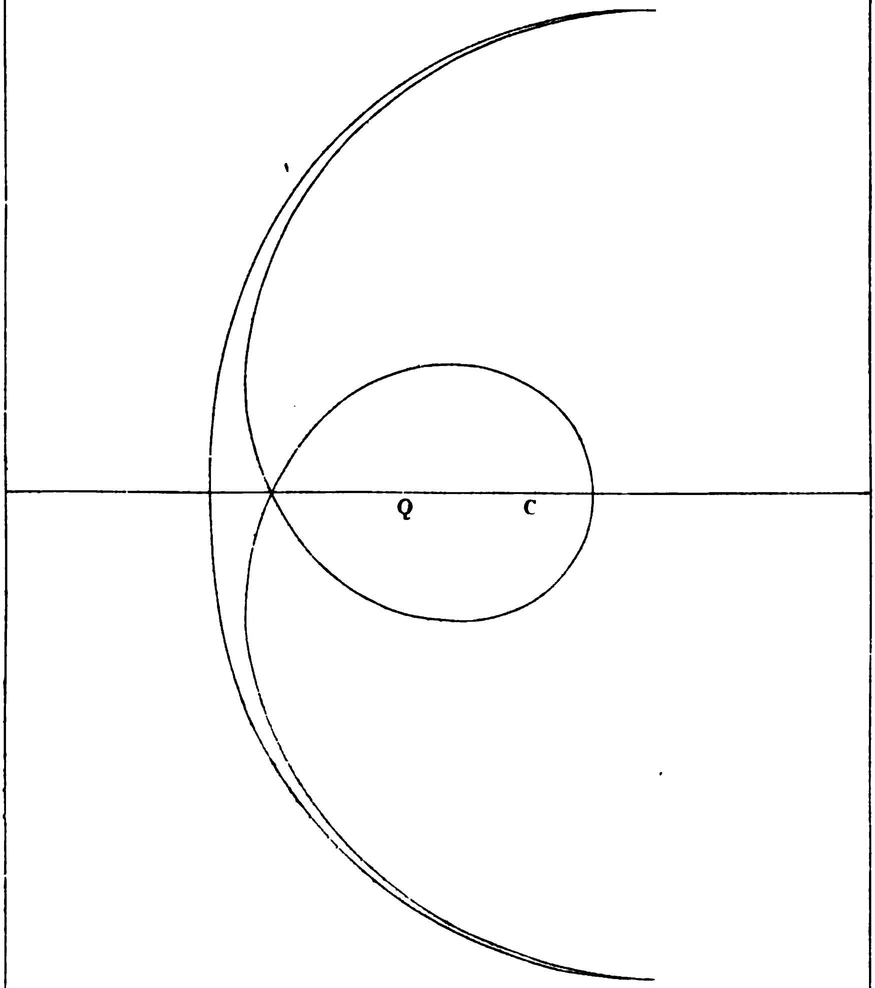
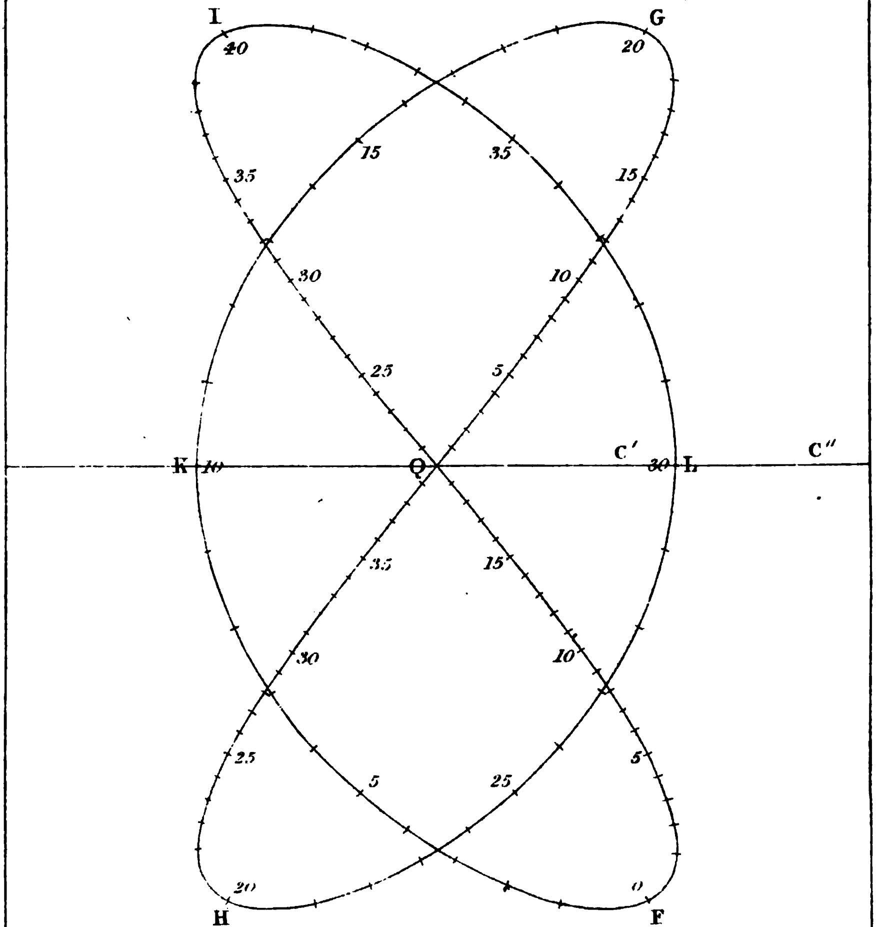


Fig. 56.

Kemend.



3 Contacts.

G.A.Amman sc.

Fig. 57. a.

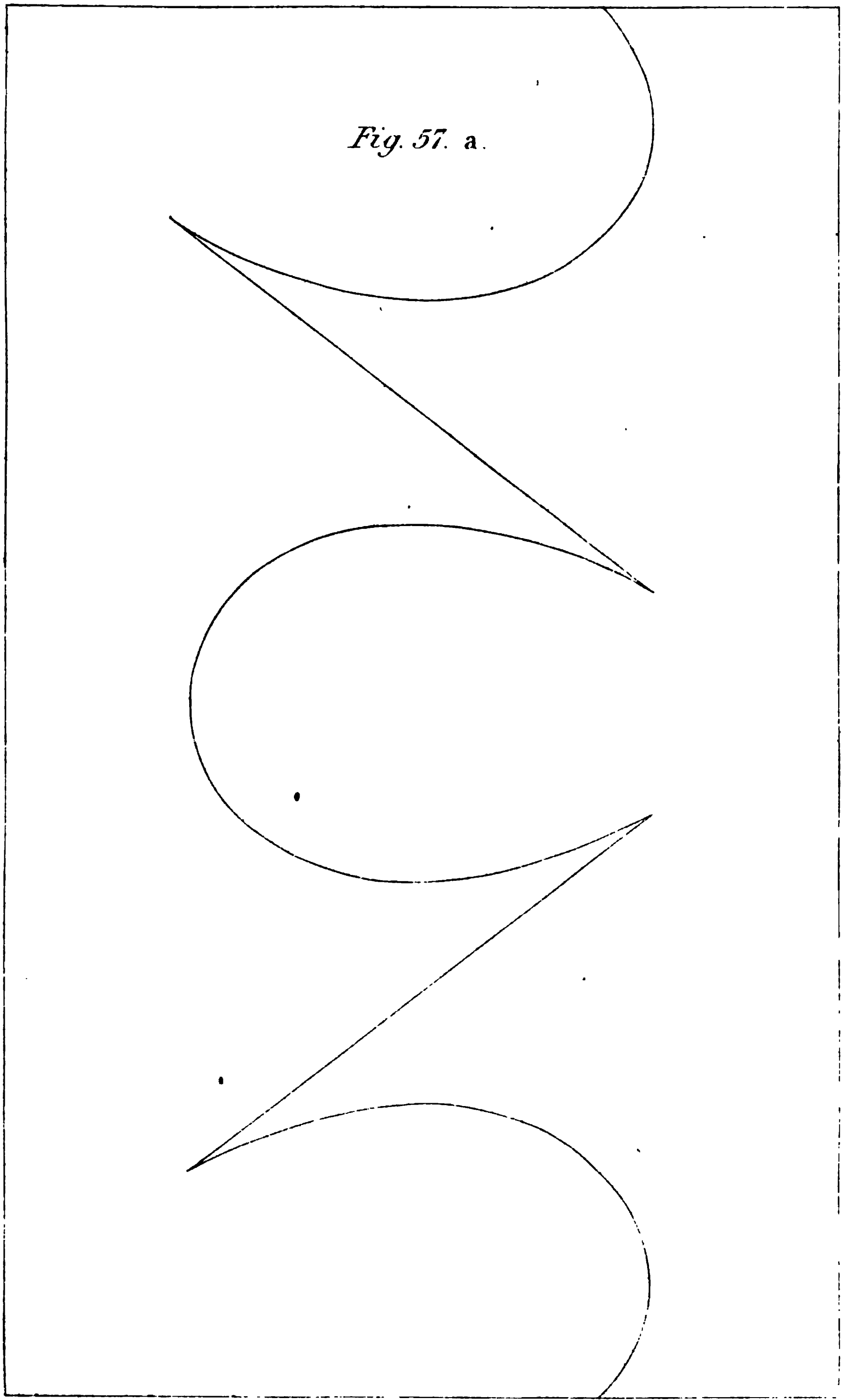
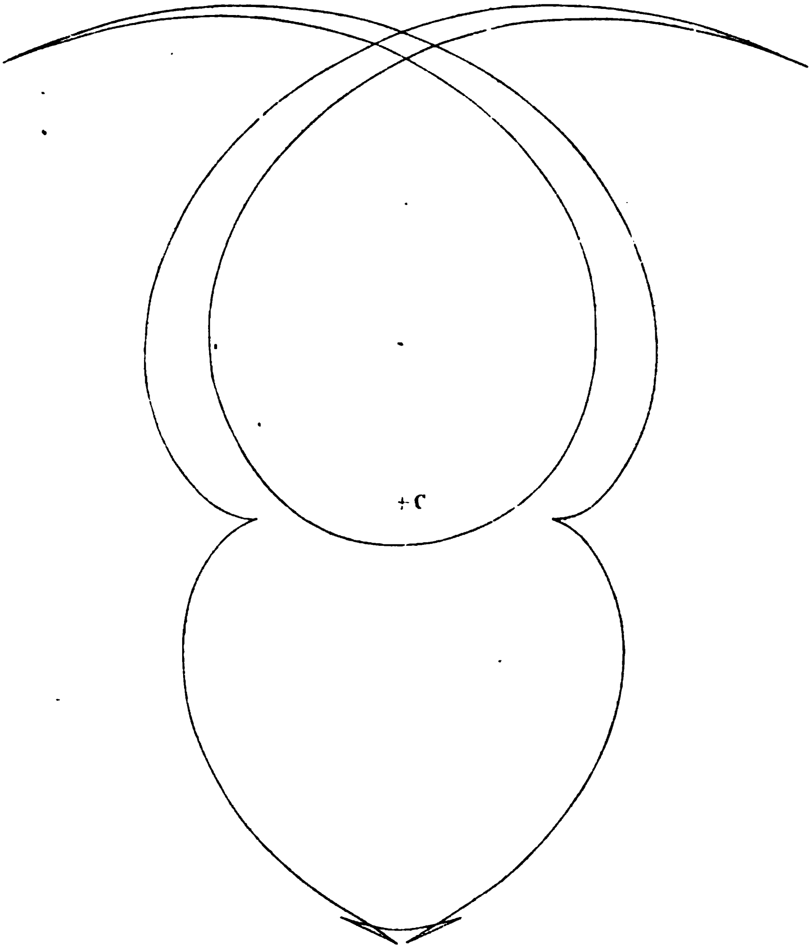


Fig. 57. b.



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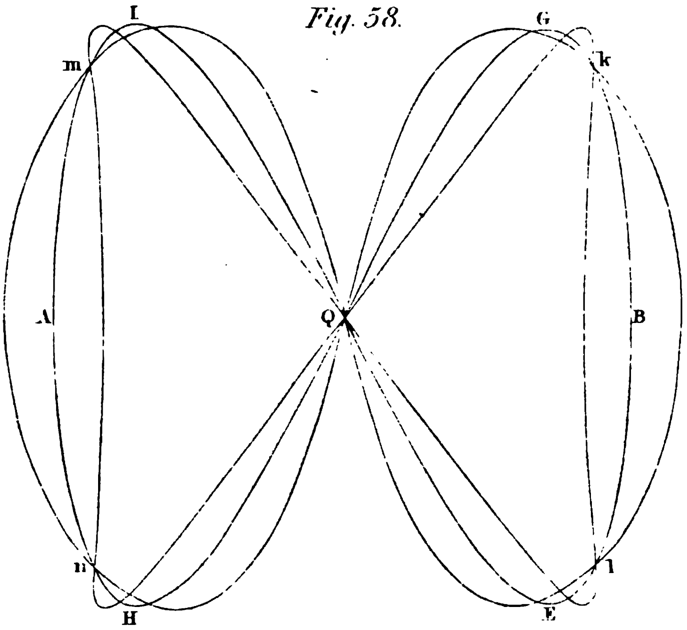

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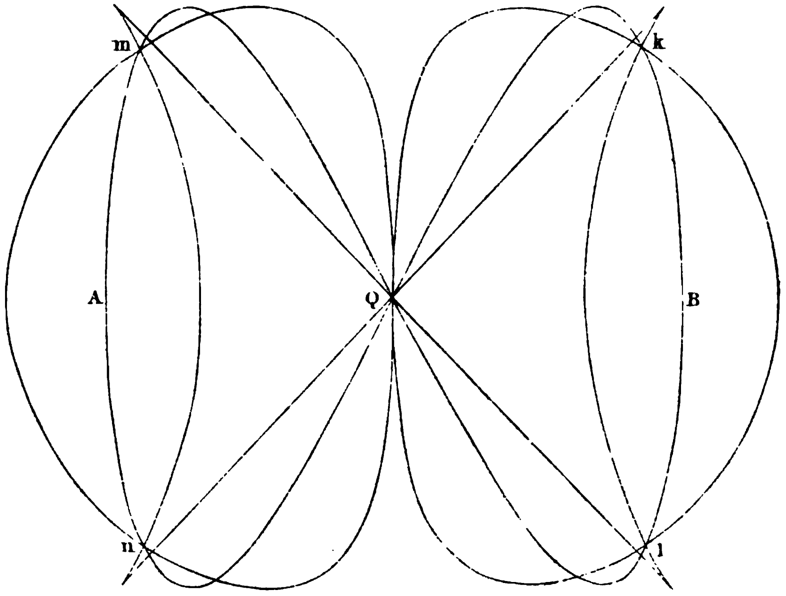

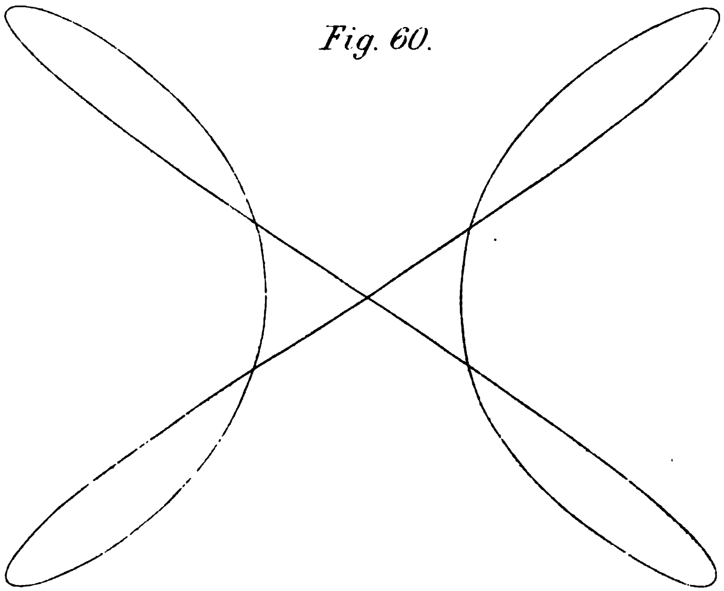

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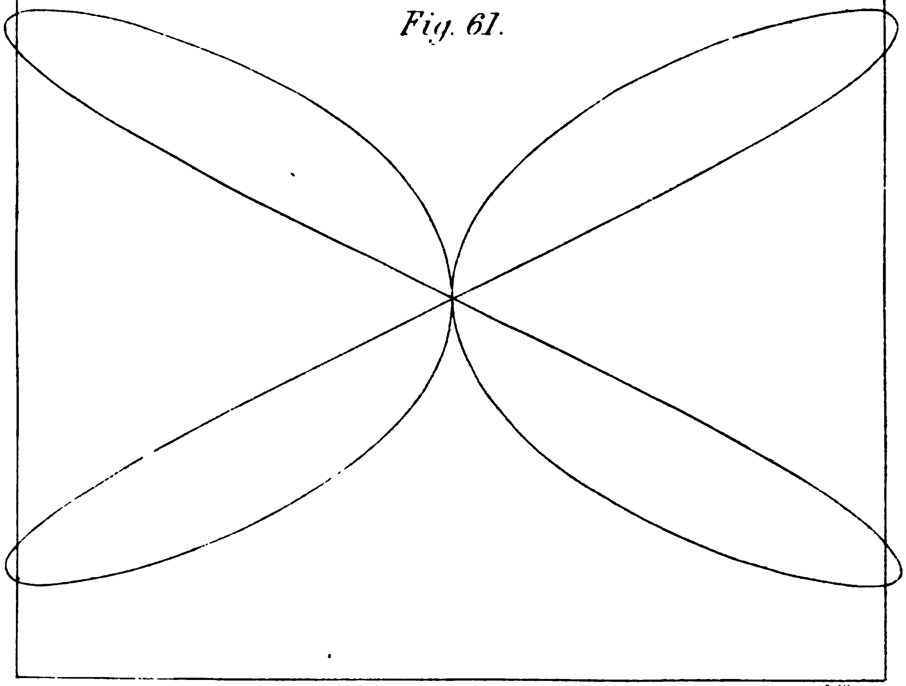

Fig. 61.

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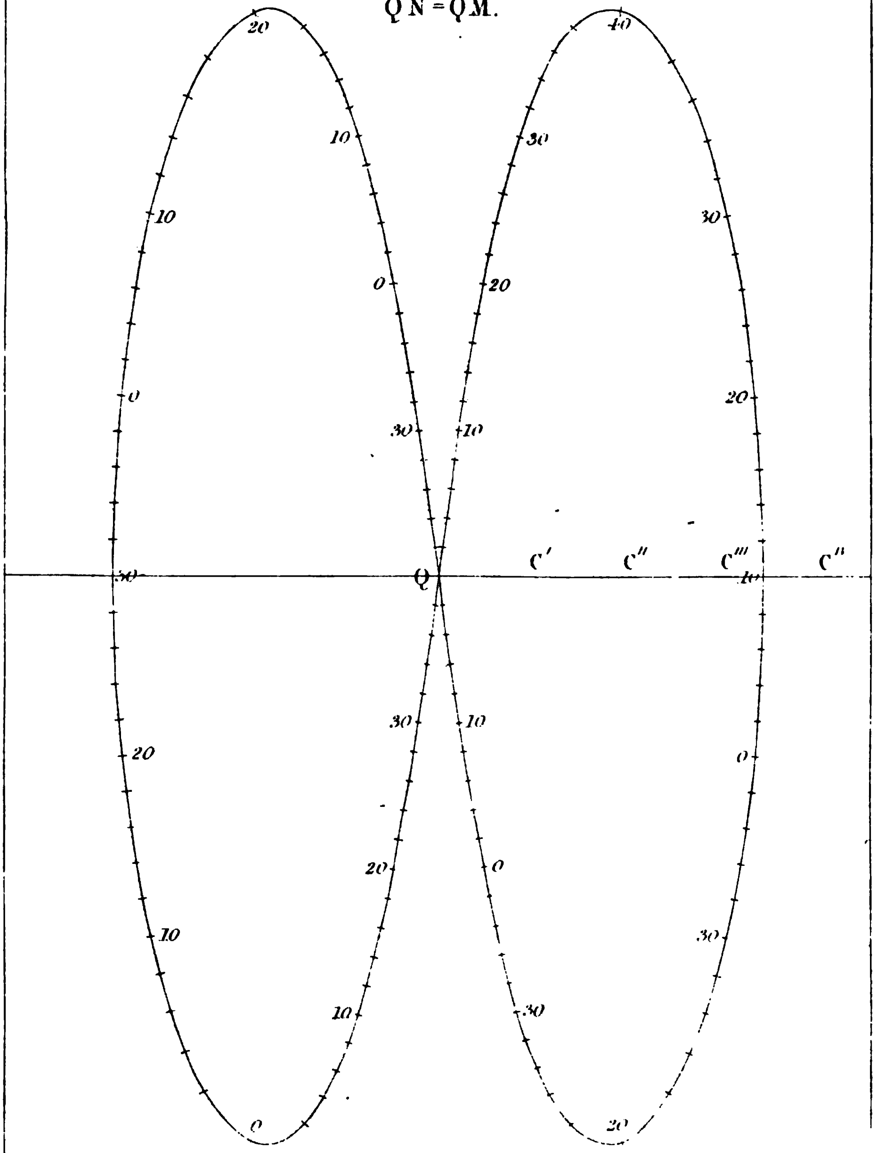
Fig. 62.

Modified Hour-glass Curve,

QN = QM.



7 Contacts.

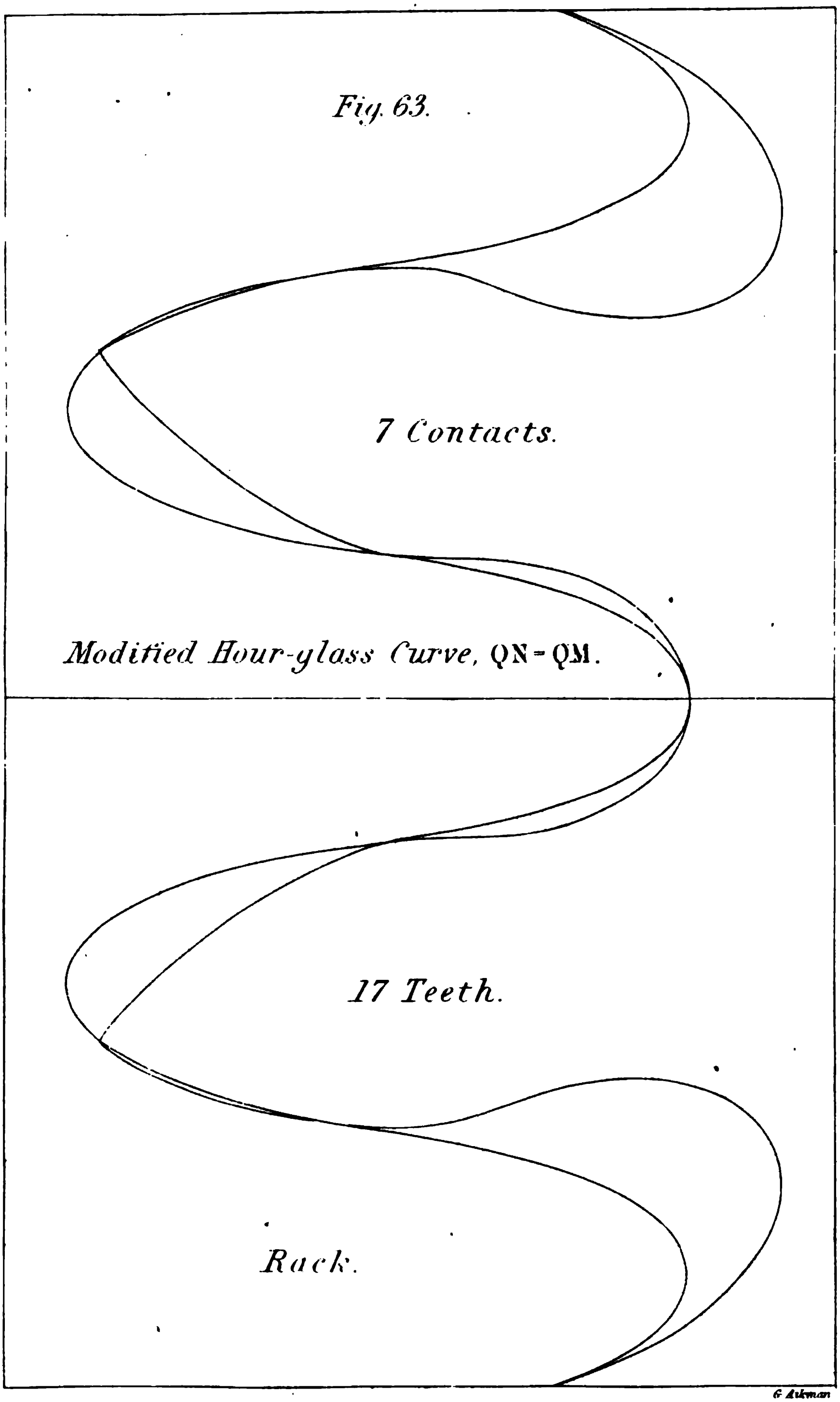


Fig. 63.

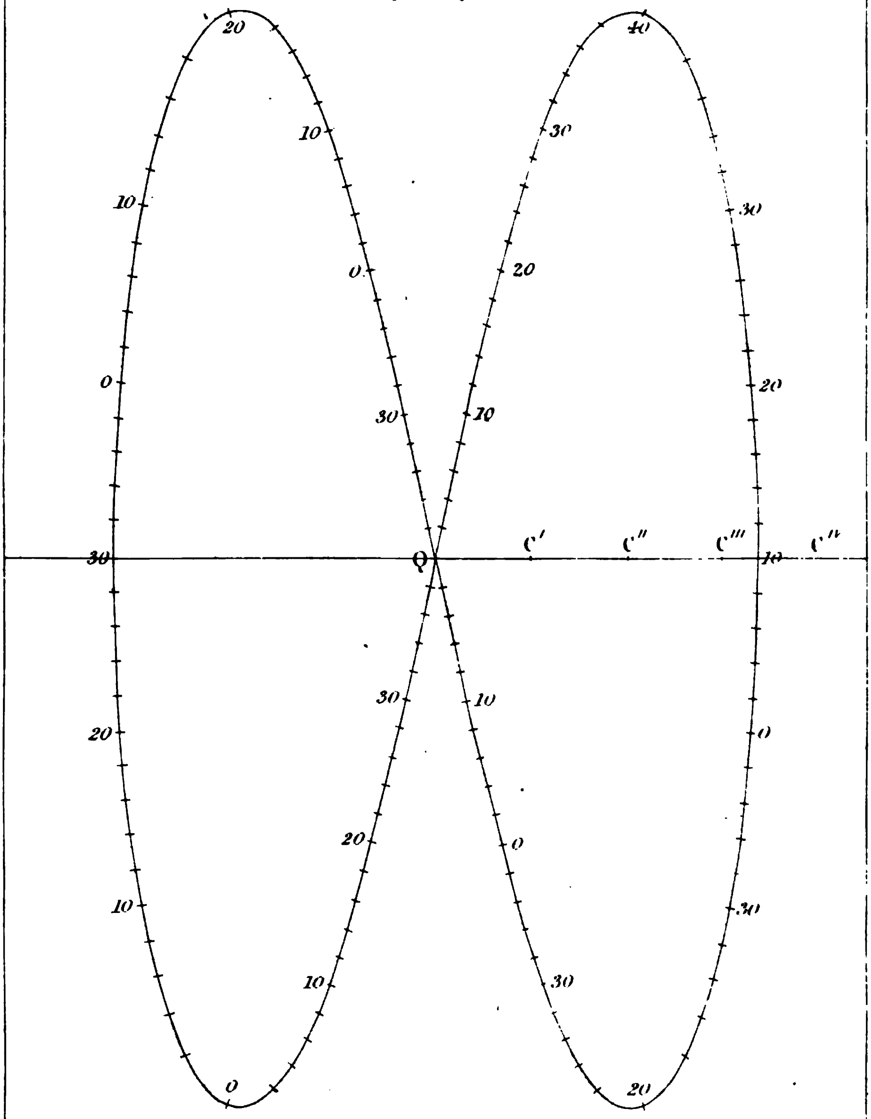
Modified Hour-glass Curve, QN = QM.

G Aikman

Fig. 64.

Modified Hour-glass Curve.

QN = QO.



7 Contacts.

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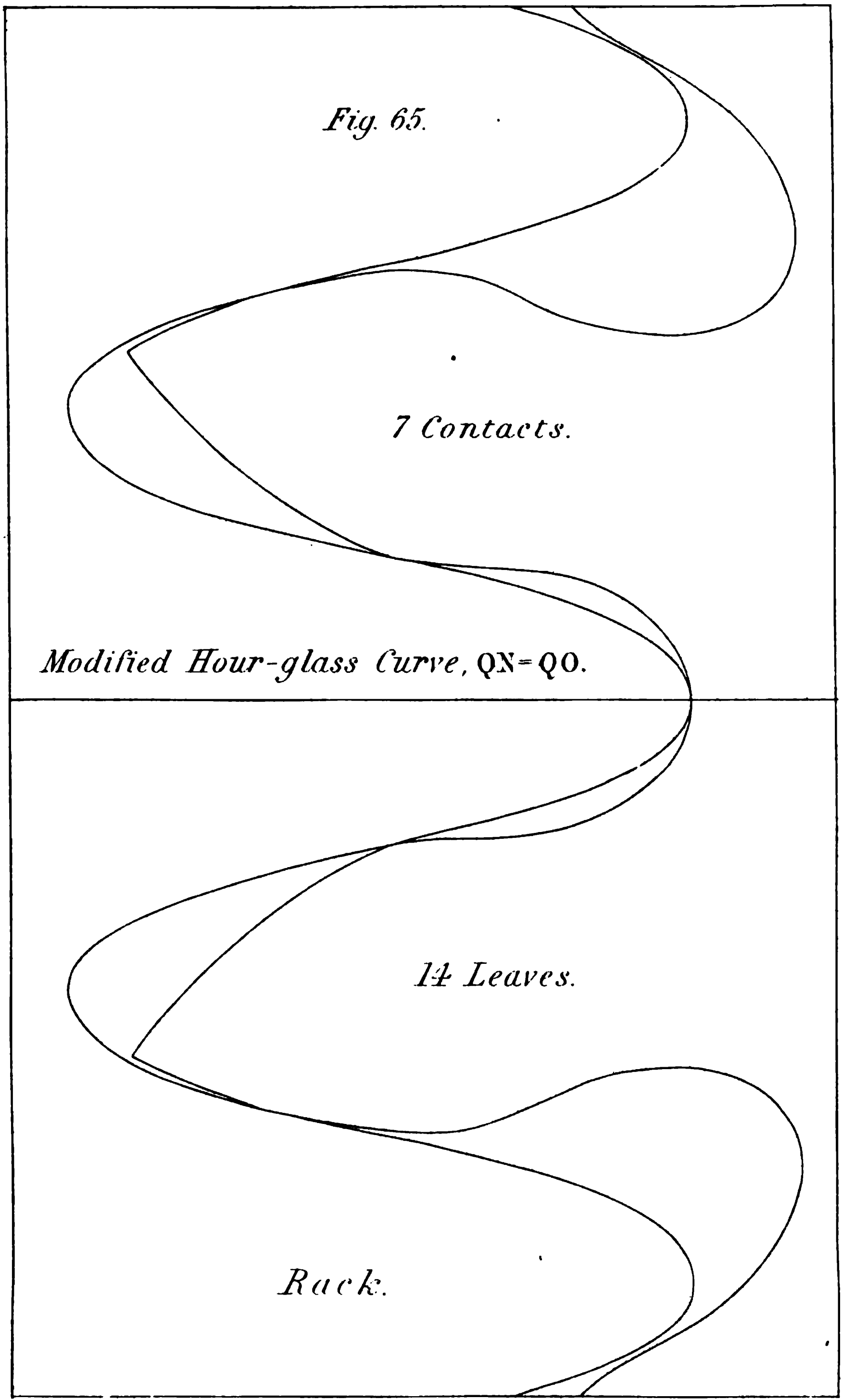


Fig. 65.

Modified Hour-glass Curve, QN = QO.

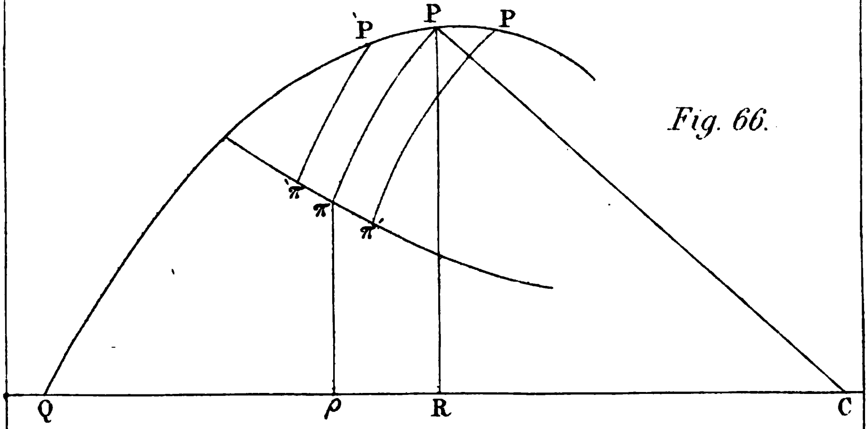


Fig. 66.

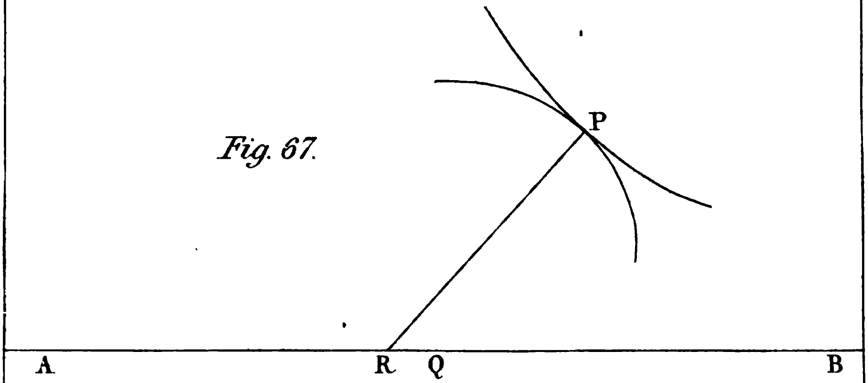
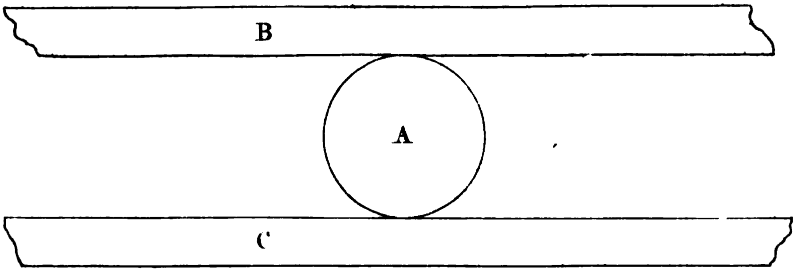


Fig. 67.

Fig. 68.



G. Aikman

Fig. 69.

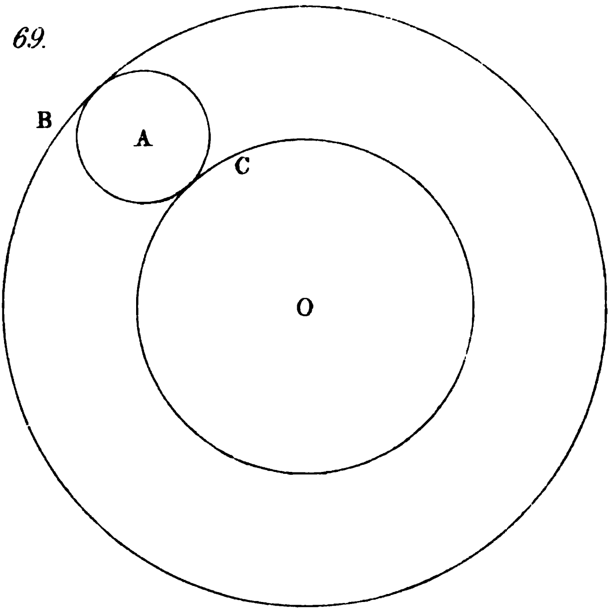
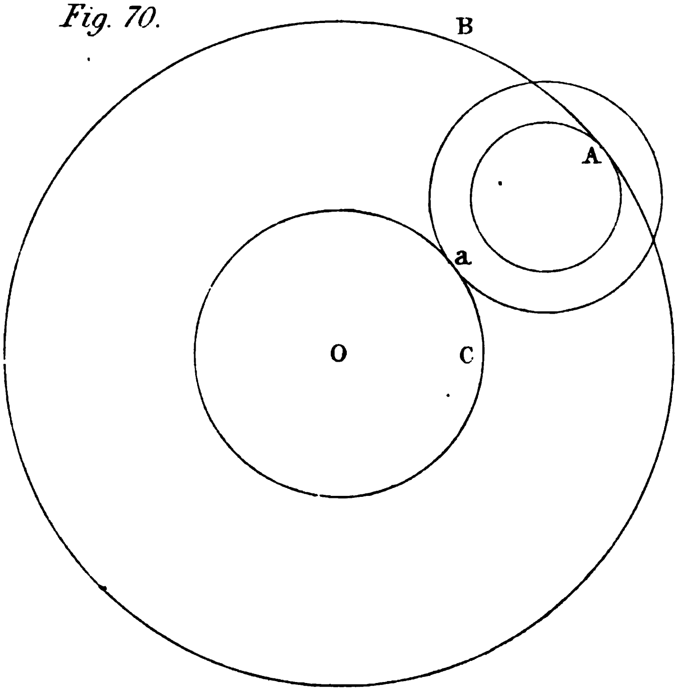


Fig. 70.



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